To facilitate the reviewing process, reference [8] is included below:

Proof of the Theorem

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Successive Packing Approach to Multidimensional Interleaving

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Firstly, the procedure of the successive packing, the concepts of the K-equivalent elements, the K-equivalent bursts are introduced. Secondly, we prove Lemma 0.1 on the shift-invariance of bursts under consideration. At the last, the main results presented in Theorem 0.1 is proved using Lemma 0.1.

Procedure 0.1. *The 2-D interleaving using the successive packing proceeds as follows. Consider a 2-D array of* $2^n \times 2^n$.

When $n = 0$, *the interleaved array of* 1×1 *is the original array itself. That is,*

$$S_1 = [s_0] \tag{1}$$

where s_0 *represents the element in the array, and* S_1 *the interleaved array with its subscript representing the total number of elements in it. In general, for a given* n, *the interleaved array is denoted by* $S_{2^{2n}}$.

The procedure is carried out successively. That is, for a given interleaved array S_i, *the interleaved array of* S_{4i} *can be generated according to*

$$S_{4i} = \begin{bmatrix} 4 \times S_i + 0 & 4 \times S_i + 2 \\ 4 \times S_i + 3 & 4 \times S_i + 1 \end{bmatrix} \tag{2}$$

where the notation of $4 \times S_i + k$ $(k = 0, 1, 2, 3)$ *represents a 2-D array that is generated from* S_i. *This means that* $4 \times S_i + k$ *has the same dimensionality as* S_i. *Furthermore, each element in* $4 \times S_i + k$ *is indexed in such a way that its subscript equals to the four times of that of the corresponding (having the same position) element in* S_i *plus* k.

Definition 0.1. *Consider a 2-D array of* $2^n \times 2^n$*, the subscripts of all of whose elements form a 1-D sequence in some fashion. Partition the array into* 2^m *blocks with the integer* m *satisfying* $1 \leq m \leq 2n-1$*. Each block thus generated contains* $K = 2^{2n-m}$ *elements. That is, each block is of size* $K = 2^{2n-m}$*. The first block contains elements having the subscripts from 0 to* $2^{2n-m}-1$*. In general, for any element having its subscript* k *satisfying* $2^{2n-m}(d-1) \leq k < 2^{2n-m}d$*,* $d \leq 2^m$*, we say that this element belongs to the dth block. All the elements belonging to the same block are said to be* K*-equivalent to one another, and form a* K*-equivalent element class.*

Definition 0.2. *Consider two bursts,* B_1 *and* B_2*, in an interleaved 2-D array. If these two bursts have the same size and shape, and each element in a burst (say,* B_1*) is either an element of another burst (say,* B_2*) or a K-equivalent element of an element of another burst (say,* B_2*), and vice versa, then we call these two bursts,* B_1 *and* B_2*, the K-equivalent bursts.*

Dealing with burst error correction, we may consider each block defined in Definition 0.1 as a 2-D codeword. A burst in the interleaved array is said to be corrected if each element in the burst has been spread in the de-interleaved array into a distinct codeword. From this point of view, it is easy to see that given two equivalent bursts, if one can be interleaved and corrected, then the other can also be interleaved and corrected.

Prior to present our major results, contained in Theorem 0.1, we need to provide the following lemma. With this powerful lemma, the proof of the major results will be straightforward.

Lemma 0.1. *Let* A *be an array of* $2^n \times 2^n$ *interleaved by using the successive packing procedure. Then, any bursts of* $2^k \times 2^k$ *in* A*, with the integer* k *satisfying* $1 \leq k < n$*, are the* K_1*-equivalent to one another, where* $K_1 = 2^{2n-2k}$*; and any bursts of* $2^k \times 2^{k+1}$ *or* $2^{k+1} \times 2^k$ *in* A*, with the integer* k *satisfying* $0 \leq k < n$*, are the* K_2*-equivalent to one another, where* $K_2 = 2^{2n-2k-1}$*.*

Proof. We prove the Lemma for the square bursts first followed by a proof for the rectangular bursts. In either cases, the mathematical induction method is employed. Since in either case the proof is trivial as $n = 0, 1$, and is very simple as $n = 2$, we start the proof as $n = 3$ in either case.

Part I. In this part, we prove the Lemma for any square bursts defined in Lemma 0.1.

I1. When $n = 3$, A is of 8×8, as shown in Figure 1. $k < n$ implies that $k = 0, 1, 2$. Since the proof for $k = 0$ is trivial, we only prove the Lemma for $k = 1, 2$.

I2. Consider $k = 2$. There are 16 elements (code symbols) in each of this type of square bursts. To correct this type of the bursts with a one-random-error-correcting code, the size of the codewords is 4. Look at the burst of 4×4, located at the top-left quadrant of A. In particular, look at the very north-west corner element, s_0. According to Definition 0.1, its 4-equivalent elements are:

s_1, s_2, s_3, which are marked in Figure 1. Similarly, each element in the burst has its own 4-equivalent elements, which follow the same distribution pattern as the 4-equivalent elements of s_0. Therefore, if we consider that the burst of 4×4, located at the top-left quadrant of A, is shifted toward the right by one column, we observe that the burst and its shifted version have three columns in common, while the first column in the burst (consisting of the elements $s_0, s_{48}, s_{12}, s_{60}$) has been replaced by the first column in the top-right quadrant (consisting of $s_2, s_{50}, s_{14}, s_{62}$). It is easy to see that each element in the first column of the burst is 4-equivalent to an element in the first column in the top-right quadrant. Hence, these two bursts are 4-equivalent according to Definition 0.2.

I3. This reasoning can be used for three more times (each time the burst of 4×4 is shifted towards the right by one column) and we can then conclude that the 4×4 burst located in the top-left quadrant is equivalent to each of its right-shifted versions. In arriving at the above, we used the transitive property of the equivalence relation [gonzalez 1992].

I4. Applying the reasoning, contained in I2 and I3, accordingly, we arrive at that the burst of 4×4, located in the top-left quadrant in A, is 4-equivalent to each of its down-shifted versions. After applying the same reasoning, contained in I2 and I3, to the burst of 4×4, located in the bottom-left quadrant in A, in considering its equivalence with its right-shifted versions, we conclude that any bursts of 4×4 in A are 4-equivalent to each other. Thus we conclude the proof of the Lemma for any square bursts when $k = 2$.

I5. Consider $k = 1$. There are 4 elements in each of this type of square bursts. To correct this type of burst errors with a one-random-error-correcting code, the size of the codewords is 16. Look at the very north-west 2×2 burst in A. In particular, the very north-west corner element s_0. Figure 1 shows all its 16-equivalent elements. Applying the reasoning used in the proof for $k = 2$ accordingly leads to the completion of the proof for $k = 1$.

I6. Having finished the proof for $n = 3$, we assume now that the Lemma holds for any square bursts in the A when $n = N$. (Hence, A may be denoted by $S_{2^{2N}}$.) To complete the proof for the Lemma to hold for any square bursts, according to the mathematical induction, we only need to show that the Lemma holds for any square bursts in A when $n = N + 1$. (Note that at this time A may be denoted by $S_{2^{2N+2}}$.)

I7. To complete the proof for square bursts as $n = N + 1$, we prove the Lemma first for $k < N$, then for $k = N$.

I8. Consider $k < N$. When $n = N$, each element in the very north-west square burst of $2^k \times 2^k$ has $2^{2N-2k} - 1$ equivalent elements in $S_{2^{2N}}$ due to the assumption made in I6. When $S_{2^{2N+2}}$ is successively packed by using $S_{2^{2N}}$ according to Equation 2, the total number of the equivalent elements of each element in the very north-west square burst of $2^k \times 2^k$, located in the top-left quadrant of $S_{2^{2N+2}}$, has increased by four times. Meanwhile, the difference in the subscripts of any two equivalent elements in the top-left quadrant of $s_{2^{2N+2}}$ has increased by four time compared with that in the $s_{2^{2N}}$. The increased equivalent elements are distributed evenly among the other three quadrants of $s_{2^{2N+2}}$ due to the successive packing (refer to Equation 2). It is therefore easy to see that the equivalent elements of any specific elements in the very north-west square burst of $2^k \times 2^k$ occupy the same positions in each of the four quadrants of $s_{2^{2N+2}}$. Furthermore, these positions are the same as that in the $s_{2^{2N}}$.

This observation can be further illustrated as follows. In Figure 2, the circled elements are all the 64−equivalent elements of the very north-west corner element, s_0, in S_{256}. That is, $n = 4$ and $k = 1$. It is noticed that the distribution pattern of the 64-equivalent elements in each of the four quadrants are the same. It is also noticed that the distribution pattern, say, in the top-left quadrant of S_{256} is the same as the 16-equivalent element distribution pattern in S_{64}, i.e., $n = 3$ and $k = 1$ as shown in Figure 1.

Obviously, this observation is true for any elements in the very north-west burst of $2^k \times 2^k$.

Therefore, the same reasoning used to prove the case of 8×8 can be applied accordingly to complete the proof for $k < N$.

I9. When $k = N$, the total number of the equivalent elements of each elements in the square burst located in the top-left quadrant is 4. Hence the situation is similar to the square bursts of 4×4 when $n = 3$. We can use the reasoning, contained in I2, to prove that the Lemma holds for $k = n$.

I10. Up to this point we have proved the Lemma for any square bursts in A.

Part II. In this part, we prove the Lemma for any rectangular bursts defined in Lemma 0.1.

II1. Consider $n = 3$. All the six possible types of rectangular bursts described in the Lemma in this situation are: 1×2, 2×4, 4×8, 2×1, 4×2, and 8×4.

II2. Consider the rectangular bursts of 4×8. Take a look at the rectangular burst of 4×8, located at the top-half of the 8×8 array. Under the circumstances, the total number of the equivalent elements of any element in the rectangular burst is 2. The equivalent elements of the elements s_0 and s_2 are depicted in Figure 3. If two equivalent elements are linked with a line segments, it is found that these two line segments cross each other. It is noted, in general, that the line segments linking any element in the left-half of the burst of 4×8, located at the top-half of the 8×8 array, with its equivalent element will have the same orientation as that of the line segment linking s_0 and its equivalent element s_1, while the line segment linking any element in the right-half of the burst with its equivalent element has the same orientation with that linking s_2 and its equivalent element s_3. Obviously, the burst of 4×8 at the top-half in A is equivalent to its one-row-down-shifted version. Repeat this procedure and use the transitive property of the equivalence relation, we can conclude that the burst is equivalent to any of its down-shifted version. This implies that all the bursts of 4×8 are equivalent to each other.

II3. Now prove the Lemma for any bursts of 2×4. At this time, the dimensionality of the equivalence is 8. Take a look at such a burst, which is located at the very north-west corner of the interleaved array of 8×8. All the 8-equivalent elements of the element s_0, together with s_0, are shown in Figure 4 with the solid line segments linking them. It is seen that these eight elements form two zigzag patterns in the array. All the 8-equivalent elements of the element s_8, including the element s_8, are shown in Figure 4 with the dashed line segments linking them. Note that the zigzag pattern associated with s_8 has just "opposite" orientation to that with s_0. It is easy to verify that all elements in the left-half of the burst of 2×4 have an equivalent element distribution pattern similar to that of s_0, while all elements in the right-half of the burst have an equivalent element distribution pattern similar to that of s_8.

Once this feature is identified, the reasoning used in the previous proof for the square bursts when $n = 3$ can be used accordingly to finish the proof for any bursts of 2×4.

II4. Now prove for any bursts of 1×2. Under this circumstances, the dimensionality of the equivalent elements for each element in the burst is 32. Figure 5 depicts the distribution of two 32-equivalent class involving, respectively, the two elements in the burst of 1×2, located in the very north-west corner of the interleaved array of 8×8. We see again two different equivalent element distribution patters. The similar reasoning to that used above leads to the completion of the proof for any bursts of 1×2.

II5. Clearly, the proof for any rectangular burst errors of 2×1, 4×2, and 8×4 can be proved accordingly.

II6. Thus, we complete the proof of the Lemma for any rectangular bursts when $n = 3$.

II7. Now, we assume that the Lemma holds for any rectangular bursts when $n = N$. We are going to show that the Lemma holds for any rectangular bursts when $n = N + 1$.

II8. The same reasoning as used in I7, I8 and I9 can be applied to the current circumstances accordingly to complete the proof at this part.

II9. We therefore conclude the whole proof of the Lemma.

$\square$

Now we are in a position to prove the following theorem.

Theorem 0.1. *Consider a 2-D array of* $2^n \times 2^n$*, partition it into* 2^m *blocks for an integer* m *satisfying* $1 \leq m \leq 2n - 1$ *according to Definition 0.1. When* m *is even and* $m = 2k$*, any burst* *of $2^k \times 2^k$ in the interleaved array, A, obtained by using the successive packing is spread in the de-interleaved array such that each element of the burst falls into a distinct block of size 2^{2n-2k}. When m is odd and $m = 2k+1$, any burst of $2^k \times 2^{k+1}$ or $2^{k+1} \times 2^k$ in A is spread in the de-interleaved array such that each element of the burst falls into a distinct block of size $2^{2n-2k-1}$.*

Meaning of Theorem 0.1. Theorem 0.1 states that in a 2-D array of $2^n \times 2^n$, A, interleaved with the successive packing technique, any square bursts of $2^k \times 2^k$ with $1 \leq k < n$ and any rectangular bursts of $2^k \times 2^{k+1}$ or $2^{k+1} \times 2^k$ with $0 \leq k < n$ can be spread such that each element in the burst falls into a distinct block in the de-interleaved array, where the block size, K, is 2^{2n-2k} for the burst of $2^k \times 2^k$, and $2^{2n-2k-1}$ for the bursts of $2^k \times 2^{k+1}$ or $2^{k+1} \times 2^k$. This indicates that, if a distinct code symbol is assigned to each element in the burst and all the code symbols associated with an individual K-equivalent class form a distinct codeword, then this technique guarantees that the burst error can be corrected with a one-random-error-correcting code. Furthermore, the interleaving degree equals to the size of the burst error, hence minimizing the number of codewords required in an interleaving scheme. In other words, with the successive packing technique the interleaving degree obtains the lower bound. In this sense, the successive packing interleaving technique is optimal. If a coding technique has a strong random-error-correcting capability, for instance, it can correct one error in every codeword of size two, then any burst errors of $2^{n-1} \times 2^n$ or $2^n \times 2^{n-1}$ in the $2^n \times 2^n$ interleaved array can be corrected. If a code, on the other hand, has less strong random-error-correcting capability, say, it can only correct one random error within a codeword of size 16, then with the successive packing only smaller burst errors, i.e., any bursts of $2^{n-2} \times 2^{n-2}$ in the interleaved array can be corrected.

Proof. Due to Lemma 0.1, which establishes the equivalent-invariance with respect to burst translation for the types of burst errors under consideration, we only need to prove the theorem for the types of bursts located at the very north-west corner of the 2-D interleaved array, A. (Below, these bursts are referred to as the north-west bursts for the sake of brevity.) This can be carried out as follows. Since it runs similarly to the proof of Lemma 0.1, we will make it short whenever possible.

We will prove the Theorem for the square bursts first followed by a proof for the rectangular bursts. In either cases, the mathematical induction method will be employed. Since in either case the proof is trivial as $n = 0, 1$, and is very simple as $n = 2$, we start the proof as $n = 3$ in either case.

Part I. In this part, we prove the theorem for all the north-west square bursts with the dimensionality defined in the Theorem.

I1. When $n = 3$, A is of 8×8, as shown in Figure 1. $k < n$ implies that $k = 0, 1, 2$. Since the proof for $k = 0$ is trivial, we only prove the theorem for $k = 1, 2$.

I2. Consider $k = 2$. There are 16 elements in the north-west square burst. To correct this burst with a one-random-error-correcting code, the size of the codewords is 4. Look at the very north-west corner element, s_0. According to Definition 0.1, its 4-equivalent elements are: s_1, s_2, s_3, which are marked in Figure 1 and are all outside of the square burst. Similarly, each element in the burst has its own 4-equivalent elements, which follow the same distribution pattern as the 4-equivalent elements of s_0. Therefore, we canconclude that all elements in the burst will be spread in the de-interleaved 2-D array so that there is only one code symbol, which is in error, within each single codeword. That is, the burst error can be corrected.

I3. Consider $k = 1$. There are 4 elements in the north-west square burst. To correct this burst with a random-error-correcting code, the size of the codewords is 16. Look at the very north-west corner element, s_0. Figure 1 shows all its 16-equivalent elements, which are all outside of the square burst. Similarly, each element in the burst has its own 16-equivalent elements, which follow the same distribution pattern as the 16-equivalent elements of s_0. Therefore, we can conclude that all elements in the burst will be spread in the de-interleaved 2-D array so that there is only one code symbol in error within each single codeword. That is, the burst error can be corrected.

I4. After the proof for $n = 3$, we assume now that the theorem holds for any north-west square bursts in A when $n = N$. (Hence, A may be denoted by $S_{2^{2N}}$.) That is, as $n = N$, all the north-west square bursts of size $2^k \times 2^k$ with $k < N$ can be corrected, where the corresponding codeword size becomes 2^{2N-2k}. To complete the proof of the theorem for all the north-west square bursts, according to the mathematical induction, we only need to show that the theorem holds for all the north-west square bursts in A when $n = N + 1$. (Note that at this time A may be denoted by $S_{2^{2N+2}}$.)

I5. To complete the proof for the north-west square bursts as $n = N + 1$, we prove the theorem first for $k < N$, then for $k = N$.

I6. Consider $k < N$. Look at the element located at the very north-west corner, s_0. When $n = N$, it has 2^{2N-2k} equivalent elements (including itself) in $S_{2^{2N}}$. When $S_{2^{2N+2}}$ is formed by using the successive packing with $S_{2^{2N}}$ according to Equation 2, the total number of the equivalent elements has increased by four times. Meanwhile, the difference of the subscripts of any two equivalent elements in the top-left quadrant of $s_{2^{2N+2}}$ has increased by four time compared with that in the $s_{2^{2N}}$. The increased equivalent elements are distributed evenly among the other three quadrants of $s_{2^{2N+2}}$ due to the successive packing (refer to Equation 2). It is easy to see that the equivalent elements of any specific elements in the north-west square burst of $2^k \times 2^k$ occupy the same positions within each of the four quadrants of $s_{2^{2N+2}}$. Furthermore, these positions are the same as that in the $S_{2^{2N}}$. It is hence observed that among the $2^{2N-2k+2}$ equivalent elements (including s_0) only is s_0 within the square burst of $2^k \times 2^k$ in $S_{2^{2N+2}}$.

Obviously, the above observation is also true for any elements in the north-west burst of $2^k \times 2^k$.

As mentioned above (in meaning of Theorem 0.1), all the $2^{2N-2k+2}$-equivalent elements form a codeword. The distribution pattern of all the $2^{2N-2k+2}$-equivalent elements discussed above indicates that there is only one error element in each codeword. Therefore, we finish the proof for $k < N$.

I7. When $k = N$, the total number of the equivalent elements of each element in the north-west square burst is 3. Hence the situation is similar to the square bursts of 4×4 when $n = 3$. We can use the reasoning, used in I2, to prove the theorem for $k = N$.

I8. Up to this point we have proved the theorem for all the north-west square bursts in A. The use of Lemma 0.1 at this point leads to the proof for all the square bursts in A.

Part II. In this part, we prove the theorem for all the north-west rectangular bursts with the dimensionality defined in the Theorem.

II1. Consider $n = 3$. All the six possible types of the rectangular bursts in this situation are: 1×2, 2×4, 4×8, 2×1, 4×2, and 8×4.

II2. Consider the rectangular bursts of 4×8, located at the top-half of the 8×8 array. As shown in the proof of Lemma 0.1, the dimensionality of the equivalence is 2, and the equivalent element of each element in the burst is located in the bottom-half of A. We hence conclude that the rectangular burst can be corrected.

II3. Now prove for the north-west burst of 2×4 in A. At this time, the dimensionality of the equivalent class is 8. The distribution pattern of all the 8-equivalent elements associated with each element in the burst has been described in the proof of Lemma 0.1 (refer to Figure 4). It is seen that within each of the 8-equivalent class there is only one element is in error. This indicates that the successive packing can correct the north-west burst error of 2×4.

II4. Now prove for the north-west burst of 1×2. Under the circumstances, the dimensionality of the equivalent elements for each element in the burst is 32. As shown in the proof of Lemma 0.1, the distribution pattern of all the 32-equivalent elements of the two elements in the burst has been depicted in Figure 5. The similar reasoning to that used above leads to the completion of the proof for the north-west burst of 1×2.

II5. Clearly, the proof for the north-west rectangular burst errors of 2×1, 4×2, and 8×4 can be proved accordingly.

II6. Thus, we complete the proof for all the north-west rectangular bursts when $n = 3$.

II7. Now, we assume that the theorem holds for any north-west rectangular bursts of $2^k \times 2^{k+1}$ or $2^{k+1} \times 2^k$ when $n = N$. We are going to show that the theorem holds for all the north-west rectangular bursts when $n = N + 1$.

II8. The reasoning used in I6 can be applied to the current circumstances accordingly to complete the proof when $k < N$.

II9. When $k = N$, the proof for the burst of $2^N \times 2^{N+1}$ can be conducted with the same reasoning as that used in proving the burst of 4×8 in the interleaved 2-D array of 8×8 (i.e., $n = 3, k = 2$). The proof for the burst of $2^{N+1} \times 2^N$ can be conducted accordingly.

II10. We therefore conclude by using the mathematical induction that any north-west rectangular bursts of $2^k \times 2^{k+1}$ or $2^{k+1} \times 2^k$ can be corrected. The use of Lemma 0.1 leads to that any burst errors of $2^k \times 2^{k+1}$ or $2^{k+1} \times 2^k$ in A can be corrected.

The proof of the theorem is thus completed.

□

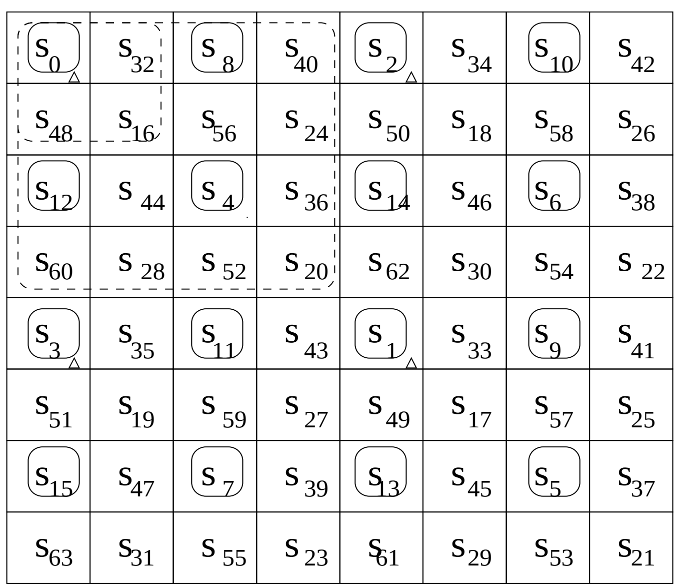

Figure 1: The 4- and 16-equivalent element distribution patterns in the interleaved 2-D array of 8×8 obtained by applying successive packing. s_0, s_1, s_2, s_3 form a 4-equivalent element class, containing s_0; $s_0, s_1, s_2, s_3, s_4, s_5, s_6, s_7, s_8, s_9, s_{10}, s_{11}, s_{12}, s_{13}, s_{14}, s_{15}$ form a 16-equivalent element class, containing s_0.

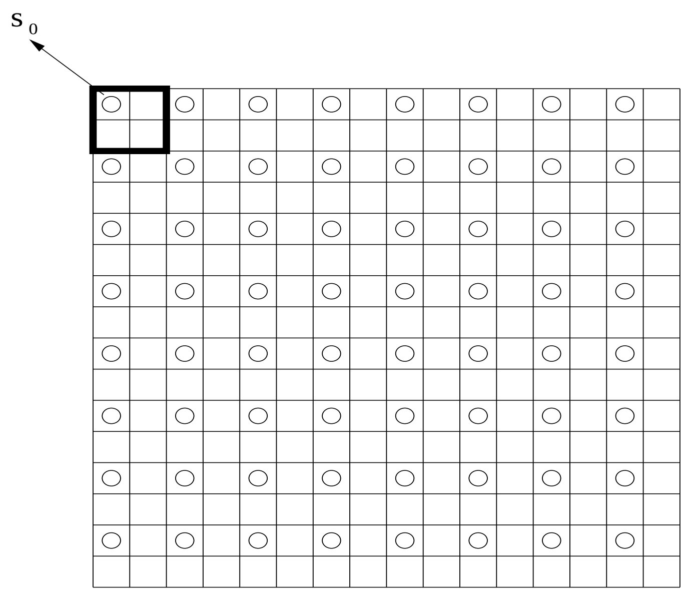

Figure 2: The 64-equivalent element distribution pattern in the interleaved 2-D array of 16 × 16 obtained by applying successive packing. The circled 64 elements form a 64-equivalent element class, which contains the very north-west element s_0.

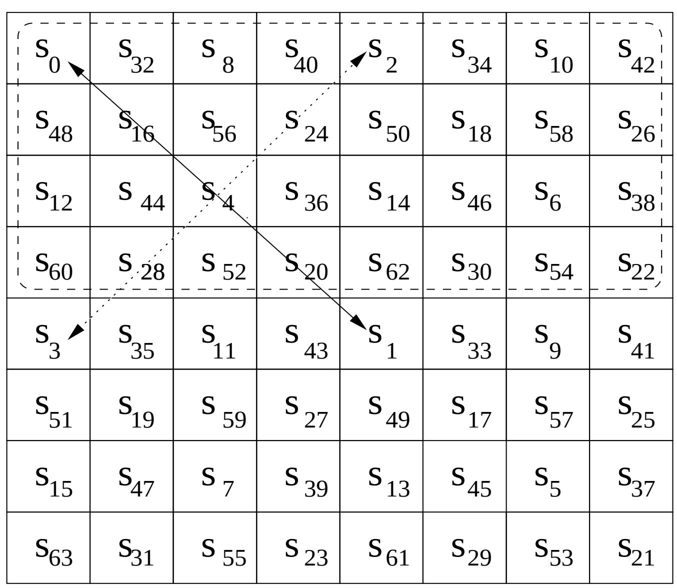

Figure 3: The 2-equivalent element distribution pattern in the interleaved 2-D array of 8 × 8 obtained by applying successive packing. The elements linked with a solid-line segment form a 2-equivalent element class containing s_0, while the elements linked with the dashed-line segment form a 2-equivalent element class containing s_2.

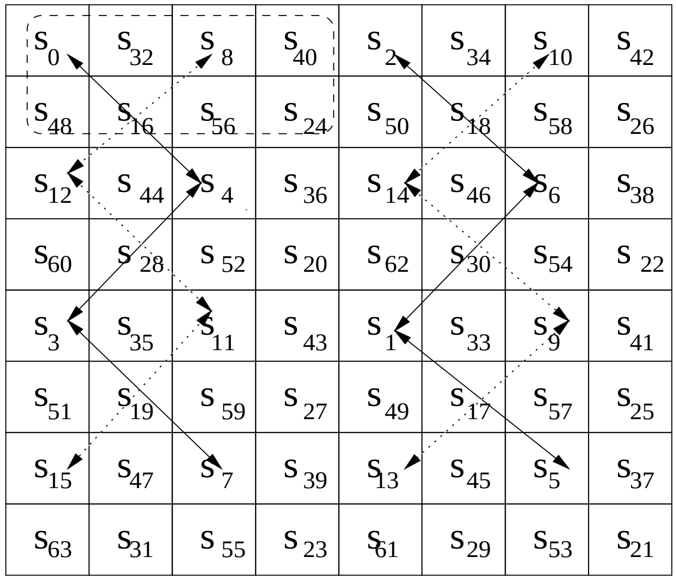

Figure 4: The 8-equivalent element distribution pattern in the interleaved 2-D array of 8×8 obtained by applying successive packing. The elements linked with the solid-line segments form a 8-equivalent element class containing s_0, while the elements linked with the dashed-line segments form a 8-equivalent element class containing s_8.

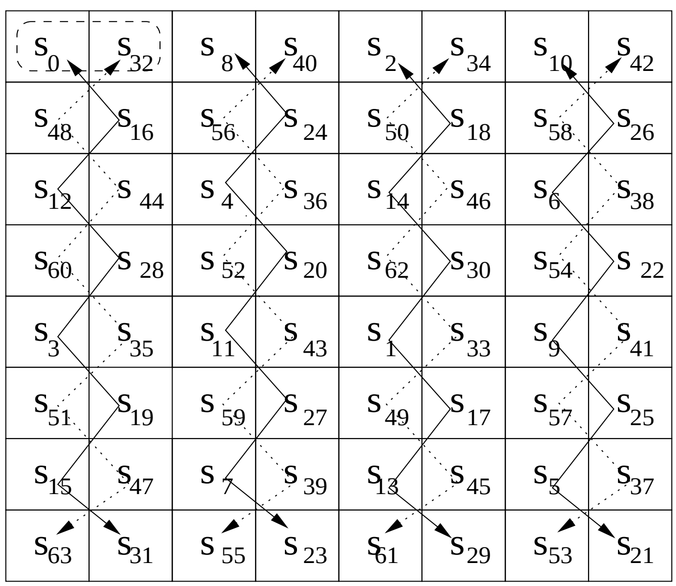

Figure 5: The 32-equivalent element distribution pattern in the interleaved 2-D array of 8×8 obtained by applying successive packing. The elements linked with the solid-line segments form a 32-equivalent element class containing s_0, while the elements linked with the dashed-line segments form a 32-equivalent element class containing s_{32}.