ECE 643 Digital Image Processing I

Chapter 4 (1st Part)

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Acknowledgement

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Preliminary Concepts

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Background

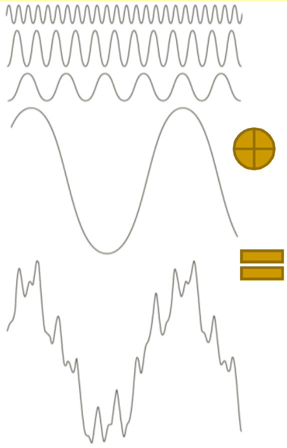

FIGURE 4.1 The function at the bottom is the sum of the four functions above it. Fourier's idea in 1807 that periodic functions could be represented as a weighted sum of sines and cosines was met with skepticism.

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Fourier Series

- A periodic function f(t) with period T

$$f(t)=\sum_{n=-\infty}^{\infty} c_n e^{j\frac{2\pi n}{T}t}$$

  - where

$$c_n=\frac{1}{T}\int_{-T/2}^{T/2} f(t)e^{-j\frac{2\pi n}{T}t}dt \qquad n\in Z$$

- Note that any complex number C = R + jI can be represented in polor coordinates as

$$C=|C|e^{j\theta}=|C|(\cos\theta+j\sin\theta) \text{ where } |C|=\sqrt{R^2+I^2},\ \theta=\arctan(I/R)$$

Impulse and Its Sifting Property

- A unit impulse of a continuous variable t

$$\delta(t)=\begin{cases}\infty & \text{if } t=0\\ 0 & \text{if } t\neq 0\end{cases} \Rightarrow \int_{-\infty}^{\infty} f(t)\delta(t)dt=f(0)$$

$$\int_{-\infty}^{\infty}\delta(t)dt=1 \qquad \int_{-\infty}^{\infty} f(t)\delta(t-t_0)dt=f(t_0)$$

- A unit discrete impulse of a discrete variable x

$$\delta(x)=\begin{cases}1 & \text{if } x=0\\ 0 & \text{if } x\neq 0\end{cases} \Rightarrow \sum_{x=-\infty}^{\infty} f(x)\delta(x)=f(0)$$

$$\sum_{x=-\infty}^{\infty}\delta(x)=1 \qquad \sum_{x=-\infty}^{\infty} f(x)\delta(x-x_0)=f(x_0)$$

Fourier Transform of Functions of One Continuous Variable

- The Fourier transform of a continuous function f(t)

$$F(u) = \Im\{f(t)\} = \int_{-\infty}^{\infty} f(t)e^{-j2\pi ut}dt = \int_{-\infty}^{\infty} f(t)[\cos(2\pi ut) - j\sin(2\pi ut)]dt$$

- The inverse Fourier transform of F(u)

$$f(t) = \int_{-\infty}^{\infty} F(u)e^{j2\pi ut}du$$

- Fourier transform pair

$$f(t) \Leftrightarrow F(u)$$

Convolution

- The convolution of f(t) and h(t)

$$f(t) * h(t) = \int_{-\infty}^{\infty} f(\tau)h(t-\tau)d\tau$$

- Convolution Theorem

$$f(t) * h(t) \Leftrightarrow H(u)F(u)$$

$$f(t)h(t) \Leftrightarrow H(u) * F(u)$$

Sampling and the Fourier Transform of Sampled Functions

Impulse Train

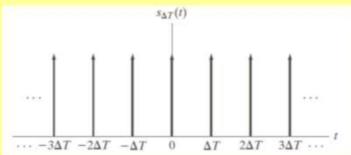


FIGURE 4.3 An impulse train.

$$s_{\Delta T}(t) = \sum_{n=-\infty}^{\infty} \delta(t - n\Delta T)$$

- An impulse train is defined as the sum of infinitely many periodic impulses, ΔT units apart

$$s_{\Delta T}(t) = \sum_{n=-\infty}^{\infty} c_n e^{j\frac{2\pi n}{\Delta T}t} = \frac{1}{\Delta T}\sum_{n=-\infty}^{\infty} e^{j\frac{2\pi n}{\Delta T}t}$$

Note that $c_n = \frac{1}{\Delta T}\int_{-\Delta T/2}^{\Delta T/2} \delta(t) e^{-j\frac{2\pi n}{\Delta T}t} dt = \frac{1}{\Delta T} e^{0} = \frac{1}{\Delta T}$

Fourier Transform of Periodic Impulse Train

$$S(u) = \Im\{s_{\Delta T}(t)\}$$

$$= \Im\left\{\frac{1}{\Delta T}\sum_{n=-\infty}^{\infty} e^{j\frac{2\pi n}{\Delta T}t}\right\}$$

$$= \frac{1}{\Delta T}\Im\left\{\sum_{n=-\infty}^{\infty} e^{j\frac{2\pi n}{\Delta T}t}\right\} = \frac{1}{\Delta T}\sum_{n=-\infty}^{\infty}\delta\left(u - \frac{n}{\Delta T}\right)$$

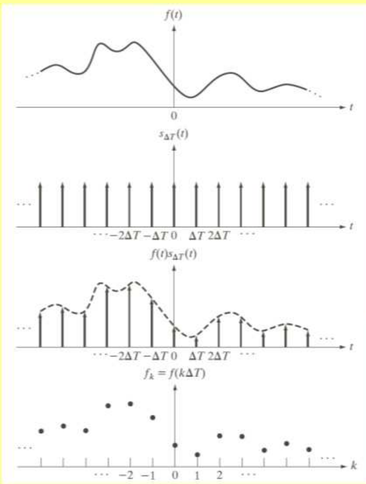
Sampling

Illustration of Sampling Process

$f(t)$ sampled into $\tilde{f}(t)$

$$\tilde{f}(t) = f(t)s_{\Delta T}(t) = \sum_{n=-\infty}^{\infty} f(t)\delta(t - n\Delta T)$$

$$\tilde{F}(u) = F(u) * S(u) = \frac{1}{\Delta T}F\left(u - \frac{n}{\Delta T}\right)$$



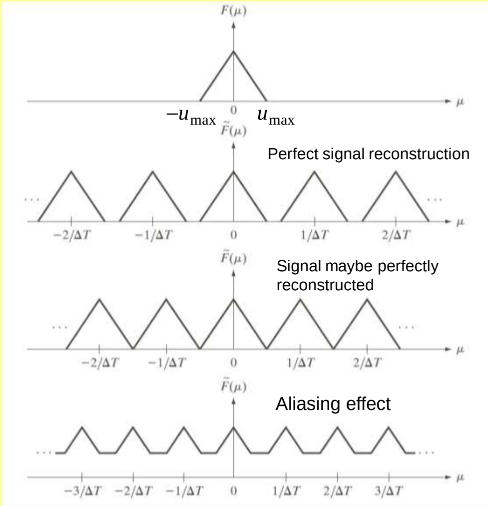
a
b
c
d

FIGURE 4.5 (a) A continuous function. (b) Train of impulses used to model the sampling process. (c) Sampled function formed as the product of (a) and (b). (d) Sample values obtained by integration and using the sifting property of the impulse. (The dashed line in (c) is shown for reference. It is not part of the data.)

- Sampling and quantization converts any continuous signal into a digital signal able to be processed in a computer.

Sampling Theorem

A band-limited signal



Over-sampling

$$f_s = \frac{1}{\Delta T} > 2u_{max}$$

Critically-sampling

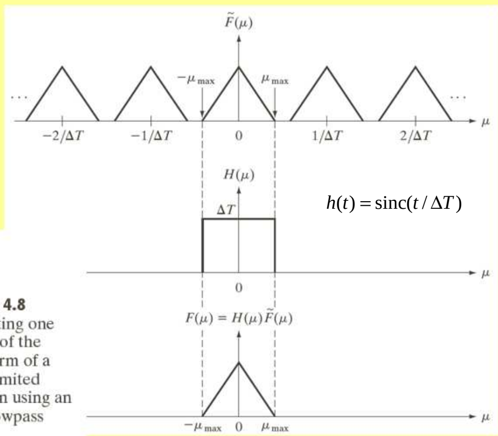
$$f_s = \frac{1}{\Delta T} = 2u_{max}$$

Under-sampling

$$f_s = \frac{1}{\Delta T} < 2u_{max}$$

Nyquist frequency: $f_{Nyq} = 2u_{max}$

Function Reconstruction (Recovery) from Sampled Data



a
b
c

FIGURE 4.8 Extracting one period of the transform of a band-limited function using an ideal lowpass filter.

$$h(t) = \text{sinc}(t/\Delta T)$$

$$F(u) = H(u)\tilde{F}(u)$$

$$f(t) = \Im^{-1}\{H(u)\tilde{F}(u)\} = h(t) * \tilde{f}(t)$$

$$f(t) = \sum_{n=-\infty}^{\infty} f(n\Delta T)\text{sinc}[(t - n/\Delta T)/n\Delta T]$$

Between sample points, f(t) are interpolations formed by the sum of the sinc functions

Discrete Fourier Transform (DFT) of One Variable

Obtaining the DFT from the Continuous Transform of a Sampled Function

Sampled data

$$\tilde{F}(u) = \int_{-\infty}^{\infty} \tilde{f}(t) e^{-j2\pi ut} dt = \sum_{n=-\infty}^{\infty} \int_{-\infty}^{\infty} f(t)\delta(t - n\Delta T) e^{-j2\pi ut} dt = \sum_{n=-\infty}^{\infty} f(n\Delta T) e^{-j2\pi un\Delta T}$$

- To obtain M equally spaced samples taken over 1/ΔT period of $\tilde{F}(u)$

DFT

$$u = \frac{m}{M\Delta T}, m = 0,1,2,\ldots, M-1 \Leftrightarrow u\Delta T = \frac{m}{M} \Longrightarrow F_m = \sum_{n=0}^{M-1} f_n e^{-j2\pi mn/M}$$

- M samples of f(t) can be recovered by IDFT

$$f(x) = \frac{1}{M} \sum_{u=0}^{M-1} F(u) e^{j2\pi ux/M}, \quad x = 0,1,2,\ldots, M-1$$

2DFT

Periodicity

$$F(u) = F(u + kM); \; f(x) = f(x + kM)$$

$$f(x) * h(x) = \sum_{m=0}^{M-1} f(m) h(x - m)$$

Circular convolution

Relationship Between the Sampling and Frequency Intervals

- If f(x) consists of M samples of a function f(t) taken ΔT units apart, the duration of record

$$T = M\Delta T$$

- The corresponding spacing in the discrete frequency domain is

$$\Delta u = \frac{1}{M\Delta T} = \frac{1}{T}$$

- The entire frequency range spanned by the M components is

$$\Omega = M\Delta u = \frac{1}{\Delta T}$$

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Extension to Functions of Two Variables

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2-D Impulse and Its Sifting Property

- An impulse of two continuous variables

$$\delta(t,z)=\begin{cases}\infty & \text{if } t=z=0\\ 0 & \text{if otherwise}\end{cases}$$

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(t,z)\delta(t,z)dtdz = f(0,0)$$

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(t,z)\delta(t-t_0,z-z_0)dtdz = f(t_0,z_0)$$

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \delta(t,z)dtdz = 1$$

- A 2-D discrete impulse

$$\delta(x,y)=\begin{cases}1 & \text{if } x=y=0\\ 0 & \text{if otherwise}\end{cases}$$

$$\sum_{x=-\infty}^{\infty}\sum_{y=-\infty}^{\infty} f(x,y)\delta(x,y) = f(0,0)$$

$$\sum_{x=-\infty}^{\infty}\sum_{y=-\infty}^{\infty} f(x,y)\delta(x-x_0,y-y_0) = f(x_0,y_0)$$

$$\sum_{x=-\infty}^{\infty}\sum_{y=-\infty}^{\infty} \delta(x,y) = 1$$

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2-D Continuous Fourier Transform Pair

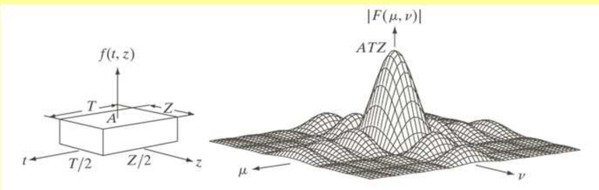

FIGURE 4.13 (a) A 2-D function, and (b) a section of its spectrum (not to scale). The block is longer along the *t*-axis, so the spectrum is more "contracted" along the *μ*-axis. Compare with Fig. 4.4.

- Forward Transform $F(u,v)=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(t,z)e^{-j2\pi(ut+vz)}dtdz$
- Backward (inverse) Transform

$$f(t,z)=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} F(u,v)e^{j2\pi(ut+vz)}dudv$$

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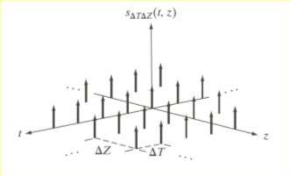
Two-Dimensional Sampling and the 2-D Sampling Theorem

- Sampling in 2-D can be modeled using 2-D impulse train



$$s_{\Delta T \Delta Z}(t,z) = \sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} \delta(t - m\Delta T, z - n\Delta Z)$$

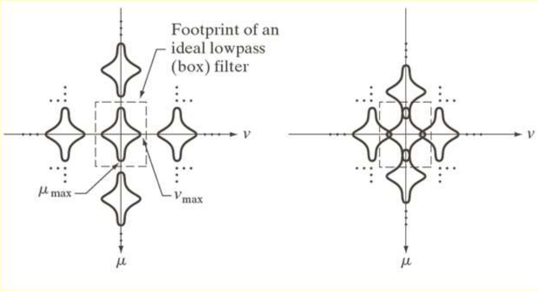
- Any band-limited 2-D signal can be perfectly recovered if

Sampling Period $\Delta T < \frac{1}{2u_{\max}}$, $\Delta Z < \frac{1}{2v_{\max}}$ ⟺ $\frac{1}{\Delta T} > 2u_{\max}$, $\frac{1}{\Delta Z} > 2v_{\max}$ Sampling Frequency

Aliasing in Images

Spatial Aliasing v.s. Temporal Aliasing

- Spatial aliasing: under-sampling (both still picture and video)
- Temporal aliasing: too low frame rate (only in video)



a b

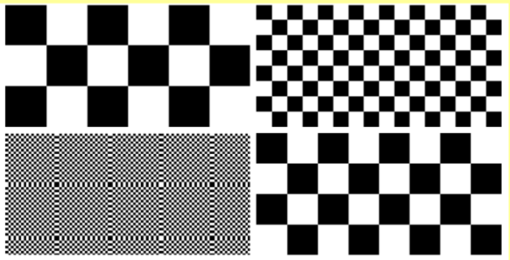
FIGURE 4.15 Two-dimensional Fourier transforms of (a) an over-sampled, and (b) under-sampled band-limited function.

Example 4.6: Aliasing in Images

- An imaging system capable of taking 96x96 pixels is to be used to digitize checkerboard patterns. It will be able to resolve pattern up to 96x96 squares, in which the size of each square, s, is 1 pixel.

s = 16 pixels
(aliasing-free)

s = 6 pixels
(aliasing-free)



s = 0.9174 pixels
(significant aliasing)

s = 0.4798 pixels appears as if s were 12 pixels
(aliasing but misleading as a normal image)

a b
c d

FIGURE 4.16 Aliasing in images. In (a) and (b), the lengths of the sides of the squares are 16 and 6 pixels, respectively, and aliasing is visually negligible. In (c) and (d), the sides of the squares are 0.9174 and 0.4798 pixels, respectively, and the results show significant aliasing. Note that (d) masquerades as a "normal" image.

Image interpolation and re-sampling

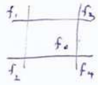
- Shrinking: under-sampling
  - The aliasing effects are generally worsened after image shrinking.
    - To reduce such effects, slightly blur an image before shrinking
      - To reduce the maximum frequencies of image spectrum
- Zooming: over-sampling
  - Creating new pixel locations
  - Assigning pixel values to these locations
- Typical Interpolation Algorithm
  - Nearest neighbor
    - Zooming: pixel replication Shrinking: pixel deletion
  - Bilinear
  - Higher order

Zooming and Shrinking

- Shrinking: under-sampling
- Zooming: oversampling/interpolation
  - Creating new pixel locations
  - Assigning gray level to these new locations
    - Nearest neighbor interpolation
    - Bilinear interpolation
    - Higher order interpolation
    - Computation complexity
    - Smoothness

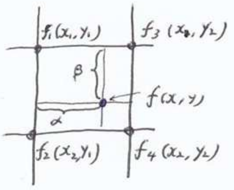
For 2) a) Nearest neighbor interpolation

(zero-order)



$f = f_4$

b) Bilinear interpolation



$0 \le \alpha, \beta \le 1$

$$f(x, y_1) = \beta f_2 + (1-\beta) f_1$$

$$f(x, y_2) = \beta f_4 + (1-\beta) f_3$$

$$f(x, y) = \alpha [\beta f_4 + (1-\beta) f_3] + (1-\alpha) [\beta f_2 + (1-\beta) f_1]$$

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c) Higher order interpolation (involving more points) ⇒ smoother results

However, more complicated computation.

Seldom used in reality.

Bilinear interpolation used often.

Fig. 2.25 Comparison btwn nearest neighbor interpolation and bilinear interpolation

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In comparison with sampling and quantization

Closely related to sampling and quantization:

zooming: oversampling (zoom in)

shrinking: undersampling (zoom out)

Difference: zooming and shrinking are applied to digital image.

Example 4.7: Illustration of Aliasing in Resampled Images

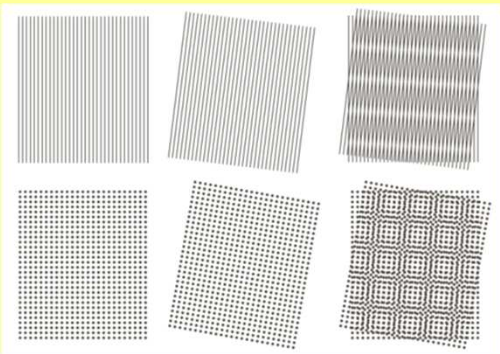
a b c

FIGURE 4.17 Illustration of aliasing on resampled images. (a) A digital image with negligible visual aliasing. (b) Result of resizing the image to 50% of its original size by pixel deletion. Aliasing is clearly visible. (c) Result of blurring the image in (a) with a 3×3 averaging filter prior to resizing. The image is slightly more blurred than (b), but aliasing is not longer objectionable. (Original image courtesy of the Signal Compression Laboratory, University of California, Santa Barbara.)

Aliasing and Moire Patterns

- When two regularly spaced sets of lines are superimposed, the appearance of a new set of lines ***(moiré pattern)*** passing through the points where the original lines cross at small angles.
- Figure 4.20
- Wikipedia: Moire

Moiré Patterns



a b c
d e f

FIGURE 4.20 Examples of the moiré effect. These are ink drawings, not digitized patterns. Superimposing one pattern on the other is equivalent mathematically to multiplying the patterns.

- Moiré patterns are more general than sampling artifacts.
- It appears to have originated with weavers who first noticed interference patterns visible on some fabrics.

Moiré Patterns

- Sometimes result from sampling scenes with periodic or nearly periodic components
- Arise routinely when scanning media print, such as newspapers and magazines, or in images with periodic components whose spacing is comparable to the spacing between samples.

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Moiré Patterns

FIGURE 4.21 A newspaper image of size 246 × 168 pixels sampled at 75 dpi showing a moiré pattern. The moiré pattern in this image is the interference pattern created between the ±45° orientation of the halftone dots and the north–south orientation of the sampling grid used to digitize the image.

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Extension to Functions of Two Variables (Continuous)

2-D Discrete Fourier Transform and Its Inverse

- Forward Transform

$$F(u,v) = \sum_{x=0}^{M-1}\sum_{y=0}^{N-1} f(x,y)e^{-j2\pi(ux/M+vy/N)}$$

- Backward Transform

$$f(x,y) = \frac{1}{MN}\sum_{u=0}^{M-1}\sum_{v=0}^{N-1} F(u,v)e^{j2\pi(ux/M+vy/N)}$$

- Relationships Between Spatial and Frequency Intervals

$$\Delta u = \frac{1}{M\Delta T} \qquad \Delta v = \frac{1}{N\Delta Z}$$

Some Properties of the 2-D Discrete Fourier Transform

- Translation and Rotation

$$f(x,y)e^{j2\pi(u_0x/M+v_0y/N)} \Leftrightarrow F(u-u_0, v-v_0)$$

$$f(x-x_0, y-y_0) \Leftrightarrow F(u,v)e^{-j2\pi(x_0u/M+y_0v/N)}$$

In the polar coordinate, $f(r,\theta+\theta_0) \Leftrightarrow F(\omega,\varphi+\theta_0)$

- Periodicity

$$F(u,v) = F(u+k_1M, v) = F(u, v+k_2N) = F(u+k_1M, v+k_2N)$$

$$f(x,y) = f(x+k_1M, y) = F(x, y+k_2N) = f(x+k_1M, y+k_2N)$$

Centering Fourier Transform

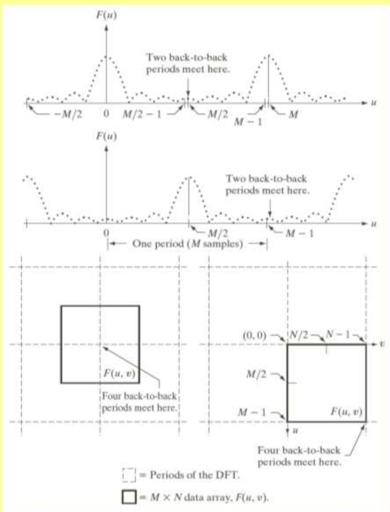
From sifting property

$$f(x)e^{j2\pi(u_0x/M)} \Leftrightarrow F(u-u_0)$$

when $u_0 = M/2$, $e^{j2\pi(u_0x/M)} = e^{(j\pi)x} = (-1)^x$

$$f(x)(-1)^x \Leftrightarrow F(u-M/2)$$

$$f(x,y)(-1)^{x+y} \Leftrightarrow F(u-M/2, v-N/2)$$



a
b
c d

FIGURE 4.23 Centering the Fourier transform. (a) A 1-D DFT showing an infinite number of periods. (b) Shifted DFT obtained by multiplying $f(x)$ by $(-1)^x$ before computing $F(u)$. (c) A 2-D DFT showing an infinite number of periods. The solid area is the $M \times N$ data array, $F(u,v)$, obtained with Eq. (4.5-15). This array consists of four quarter periods. (d) A Shifted DFT obtained by multiplying $f(x,y)$ by $(-1)^{x+y}$ before computing $F(u,v)$. The data now contains one complete, centered period, as in (b).

Symmetry Properties

- Even functions are said to be symmetric and odd functions are anti-symmetric.
- Since all indices in DFT and IDFT are positive, when we talk about symmetry (anti-symmetry) we are referring to such properties about the center point of a sequence.

$$w_e(x,y) = w_e(M-x, N-y)$$
$$w_o(x,y) = -w_o(M-x, N-y)$$

- A property used frequently in DIP is that the Fourier transform of a real function is conjugate symmetric:

$$F^*(u,v) = F(-u,-v)$$

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Fourier Spectrum and Phase Angle

- The 2-D DFT is complex in general, efficiently expressed in polar form.

$$F(u,v) = |F(u,v)|e^{j\phi(u,v)}$$

where

$$|F(u,v)| = \left[R^2(u,v) + I^2(u,v)\right]^{1/2} \qquad \phi(u,v) = \arctan\left[\frac{I(u,v)}{R(u,v)}\right]$$

the Fourier (or frequency) spectrum has even symmetry — the phase angle has odd symmetry

- Power spectrum: $P(u,v) = |F(u,v)|^2 = R^2(u,v) + I^2(u,v)$
- The DC component is proportional to the average value of the signal.

$$F(0,0) = \sum_{x=0}^{M-1}\sum_{y=0}^{N-1} f(x,y) = MN\left[\frac{1}{MN}\sum_{x=0}^{M-1}\sum_{y=0}^{N-1} f(x,y)\right] = MN\bar{f}(x,y) \Longrightarrow F(0,0) = MN\left|\bar{f}(x,y)\right|$$

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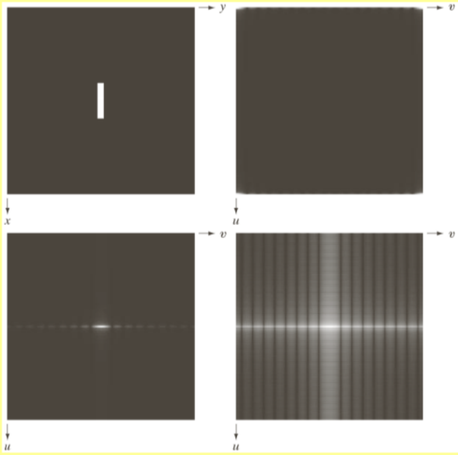
Example 4.13: The 2-D Fourier spectrum of a simple function



a b
c d

FIGURE 4.24
(a) Image. (b) Spectrum showing bright spots in the four corners. (c) Centered spectrum. (d) Result showing increased detail after a log transformation. The zero crossings of the spectrum are closer in the vertical direction because the rectangle in (a) is longer in that direction. The coordinate convention used throughout the book places the origin of the spatial and frequency domains at the top left.

END