Signals & Linear Systems Analysis Chapter 2&3, Part II

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Text used for the course: <Modern Digital and Analog Communication Systems>, 4th Edition, Lathi and Ding, Oxford

Signal and System

- Signal: A signal is defined as the time history of some quantity, usually a voltage or current.
  - Deterministic Signals
  - Random Signals
- System: A system is a combination of devices and networks (subsystems) chosen to perform a desired function.

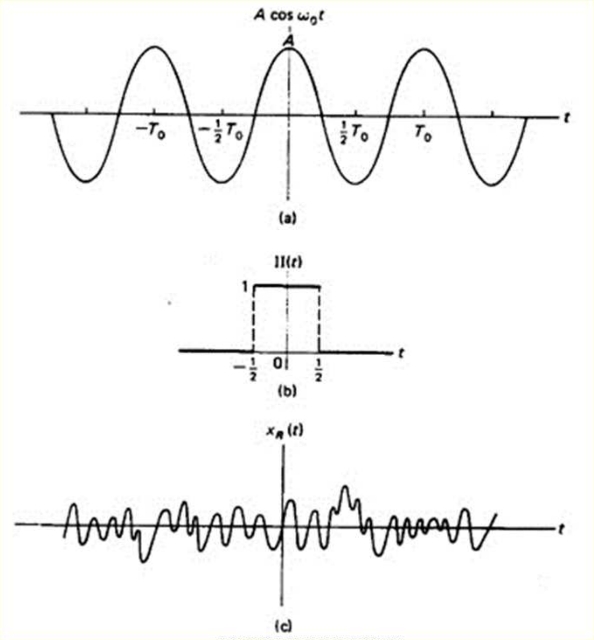
Signal Models

- ***Deterministic Signals*** are modeled as completely specified functions of time.
- Examples:
  - $x(t) = A\cos\omega_0 t, -\infty < t < \infty$ Sinusoidal
    - A: magnitude
    - ω_0: angular frequency
  - $x(t) = \begin{cases} 1 & |t| < \frac{1}{2} \\ 0 & otherwise \end{cases}$ unit rectangular pulse function denoted by $\prod(t)$

Signal Models…

- ***Random signals*** are signals that take on random values at any given time instant and must be modeled probabilistically.

- Example → Figure in the next slide
  - a) A sinusoidal signal – deterministic
  - b) Unit rectangular pulse signal – deterministic
  - c) A random signal (its one sample function)



Signal Models…

- ***Periodic signals*** A signal x(t) is periodic if
$$x(t+T_0)=x(t), -\infty<t<\infty$$
where the constant T_0 is a period. (deterministic)
- ***Fundamental period*** The smallest period is referred to as fundamental period.
- ***Aperiodic signals*** Any signal not satisfying
$$x(t+T_0)=x(t) \qquad \forall t$$
is called aperiodic.

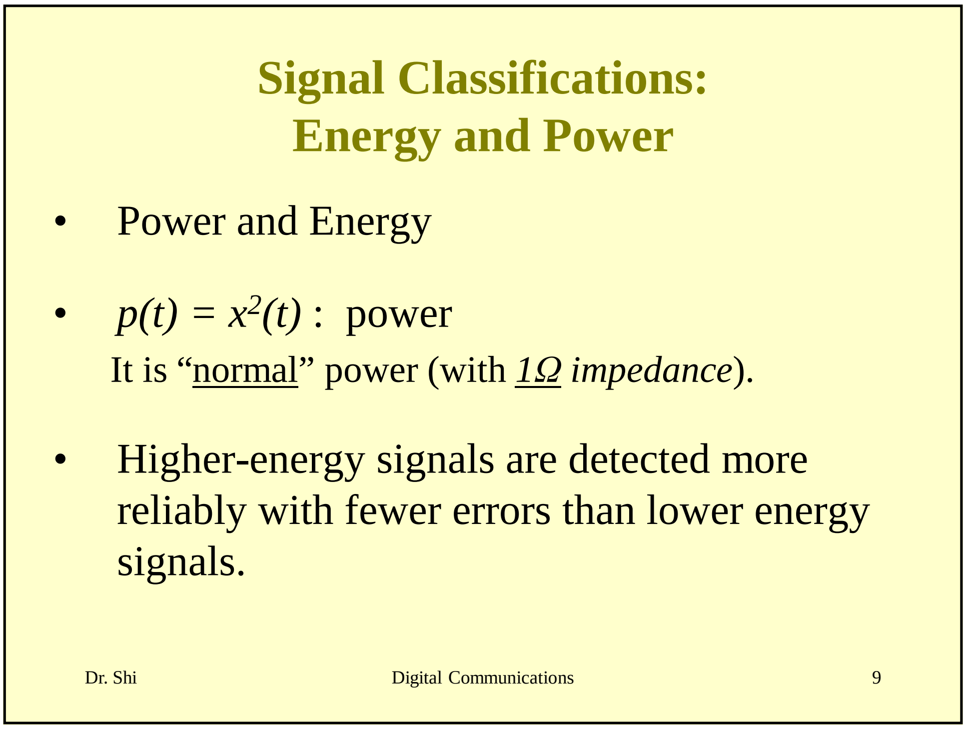
Signal Models…

- ***Rotating phasor*** A useful tool to deal with sinusoidal quantities.
  - A rotating phasor: $\tilde{x}(t) = Ae^{j(\omega_0 t+\theta)} \qquad -\infty < t < \infty$
  - Three parameters:
    - A: amplitude
      θ: phase (in radians)
      ω_0: frequency (in radians per sec)
- Phasor: $Ae^{j\theta}$ where $e^{j\omega_0 t}$ is implicit.

- The rotating phasor is periodic:

$$\tilde{x}(t) = \tilde{x}(t+T_0), \qquad T_0 = \frac{2\pi}{\omega_0}$$

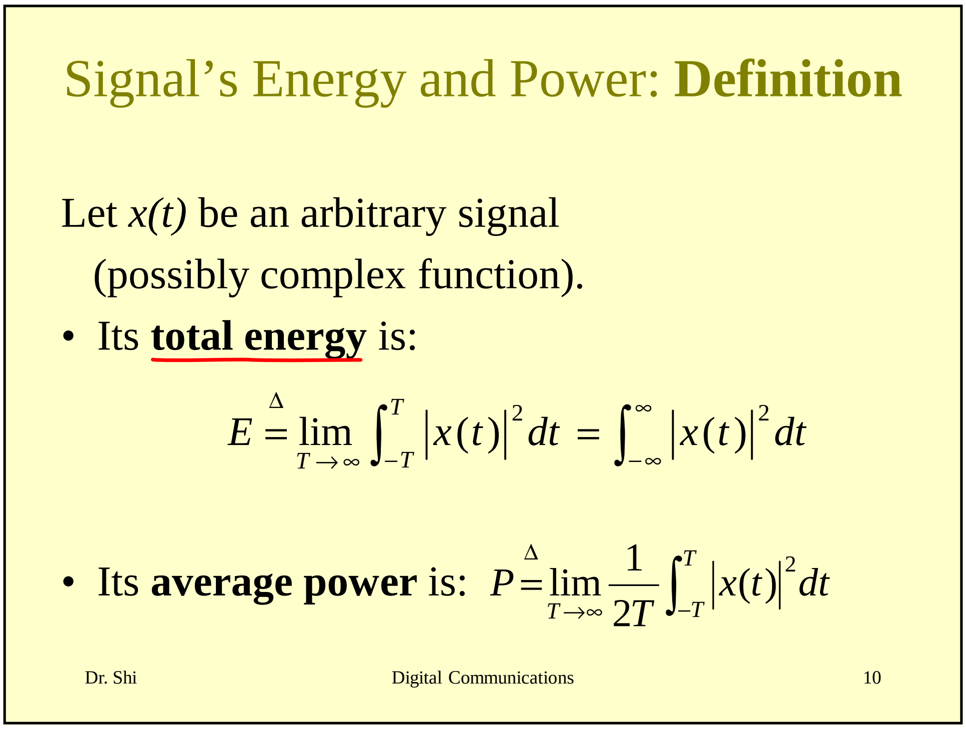
$$\begin{aligned}
\because \tilde{x}(t+T_0) &= Ae^{j[\omega_0(t+T_0)+\theta]} \\
&= A\cos\left[\omega_0(t+\frac{2\pi}{\omega_0})+\theta\right] + jA\sin\left[\omega_0(t+\frac{2\pi}{\omega_0})+\theta\right] \\
&= A[\cos(\omega_0 t+\theta) + j\sin(\omega_0 t+\theta)] \\
&= Ae^{j(\omega_0+\theta)} = \tilde{x}(t)
\end{aligned}$$



Signal Classifications: Energy and Power

- Power and Energy
- $p(t) = x^2(t)$: power

  It is "normal" power (with *1Ω impedance*).
- Higher-energy signals are detected more reliably with fewer errors than lower energy signals.



Signal's Energy and Power: **Definition**

Let *x(t)* be an arbitrary signal (possibly complex function).

- Its **total energy** is:

$$E \overset{\Delta}{=} \lim_{T\to\infty} \int_{-T}^{T} |x(t)|^2 dt = \int_{-\infty}^{\infty} |x(t)|^2 dt$$

- Its **average power** is: $P \overset{\Delta}{=} \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} |x(t)|^2 dt$

Energy signal vs. Power signal

- The function $x(t)$ is an energy signal iff

$$0 < E < \infty$$

(Hence P = 0, because of having non-zero and finite energy.)

- The function $x(t)$ is a power signal iff

$$0 < P < \infty$$

(Thus $E = \infty$, because of having non-zero and finite power.)

- For a periodic signal $x_p(t)$

$$P = \frac{1}{T_0}\int_{t_0}^{t_0+T_0} |x_p(t)|^2 \, dt \qquad T_0 \text{ is the period.}$$

- No need to carry out the limiting operation to find P for a periodic signal.
- Energy and power classifications are mutually exclusive.
  1. An energy signal must have zero average power, $\Rightarrow$ not a power signal.
  2. A power signal must have infinite energy, $\Rightarrow$ not an energy signal
- There are some signals: neither energy nor power signals. (The ramp signal is such an example.)

Example 1

- Refer to a figure about unit step function.

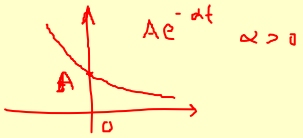
$$\alpha > 0 \quad u(t) = \begin{cases} 1 & t \geq 0 \\ 0 & t < 0 \end{cases}$$

$$x(t) = Ae^{-\alpha t}u(t)$$

$$E = \lim_{T \to \infty} \int_{-T}^{T} |x(t)|^2 dt = \int_0^\infty \frac{A^2}{e^{2\alpha t}} dt$$

$$= A^2 \left. \frac{\overset{-}{e}^{2\alpha t}}{-2\alpha} \right|_0^\infty = \frac{A^2}{2\alpha} e^{-2\alpha t} \Big|_\infty^0 = \frac{A^2}{2\alpha}$$

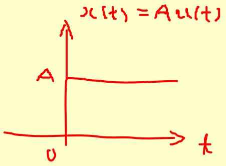
$\Rightarrow x(t)$ is an energy signal.



- If $\alpha \to 0, x(t) = Au(t)$, $\Rightarrow$ infinite energy.

$$P = \lim_{T \to \infty} \frac{1}{2T} \int_{-T}^{T} |x(t)|^2 dt = \lim_{T \to \infty} \frac{1}{2T} \int_0^T A^2 dt = \frac{1}{2} A^2$$

- Final power.
- It is a power signal



Example 2

- Refer to figure shown before.

$$x_p(t) = A\cos(\omega_0 t + \theta)$$

a sinusoidal signal
infinite energy

$$P = \frac{1}{T_0}\int_{t_0}^{t_0+T_0} A^2\cos^2(\omega_0 t + \theta)dt$$

$$= \frac{1}{T_0}\int_0^{T_0} \frac{A^2}{2}[1 + \cos 2(\omega_0 t + \theta)]dt$$

$$= \frac{A^2}{2}$$

It is a power signal.

- A frequently used skill:

$$\int_0^{T_0} \cos[2(\omega_0 t + \theta)]dt = 0$$

$$\because \frac{2\pi}{2\omega_0} = \frac{\pi}{\omega_0} = \frac{\pi}{\frac{2\pi}{T_0}} = \frac{T_0}{2}$$

- That is, period of $\cos[2(\omega_0 t + \theta)]$ becomes half.
- Integration of a sinusoid within an integral number of periods is zero.

Energy Spectral Density (ESD)

1. <u>Derivation</u>

   Total Energy:

$$1\quad E=\int_{-\infty}^{\infty}\left|\underline{x(t)}\right|^2dt$$

$$=\int_{-\infty}^{\infty}\left[\int_{-\infty}^{\infty}X(f)e^{j2\pi ft}df\right]x^*(t)dt$$

$$=\int_{-\infty}^{\infty}X(f)\left[\int_{-\infty}^{\infty}x^*(t)e^{j2\pi ft}dt\right]df$$

$$=\int_{-\infty}^{\infty}X(f)X^*(f)\overset{df}{\underset{\wedge}{=}}\int_{-\infty}^{\infty}\left|\underline{X(f)}\right|^2df=\int_{-\infty}^{\infty}\xi_x(f)df$$

ESD…

2. <u>Definition</u>

   Energy spectral density of a signal *x(t)*.

$$\xi_x(f)\overset{\Delta}{=}\left|X(f)\right|^2$$

$$\mathrm{E}=\int_{-\infty}^{\infty}\xi_x(f)df=2\int_{0}^{\infty}\xi_x(f)df$$

ESD …

3. Unit

$$X(f)=\int_{-\infty}^{\infty} x(t)e^{-j2\pi ft}dt$$

$$\left|X(f)\right|^2 : (volts-seconds)^2$$

$\because$ On a per ohm basis for total energy & average power

$\therefore$ $(volts-seconds)^2 \Rightarrow \dfrac{(volts)^2}{ohm}\cdot(seconds)^2$

$$= \frac{watts\cdot second}{hertz}, \left(\frac{1}{hertz}=second\right)$$

$$= joules/hertz$$

It is the Energy Density.

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4. Comment 1. (Parseval’s Theorem) (Rayleigh’s Energy Theorem)

$$E=\int_{-\infty}^{\infty}\left|x(t)\right|^2 dt=\int_{-\infty}^{\infty}\left|X(f)\right|^2 df$$

Comment 2.

$\xi_x(f)$ is energy spectral density.

$$\because \int_{-\infty}^{\infty}\xi_x(f)df = E$$

Power Spectral Density (PSD)

1.Consider x(t) as a real-valued power signal, then its average power is

$$P_x = \lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T} x^2(t)dt$$

This is similar to definition given before (slide 10) except that the absolute value sign has been removed.

PSD…

2. If x(t) is a periodic signal with period T_0, then its average power is

$$P_x = \frac{1}{T_0}\int_{-\frac{T_0}{2}}^{\frac{T_0}{2}} x^2(t)dt$$

$$P_x = \frac{1}{T_0}\int_{-\frac{T_0}{2}}^{\frac{T_0}{2}} x(t)\cdot x^*(t)dt$$

$$= \sum_{n=-\infty}^{\infty} \left|X_n\right|^2$$

where X_n is Fourier series (FS) coefficient of x(t).

PSD…

- This can be proved by using FS of x(t).
- Loosely speaking,

$$x(t) = \sum_{n=-\infty}^{\infty} X_n \exp\left(\frac{j2\pi nt}{T_0}\right)$$

$$x^*(t) = \sum_{n'=-\infty}^{\infty} X_{n'}^* \exp\left(-\frac{j2\pi n't}{T_0}\right)$$

All cross-product terms $\Rightarrow$ 0, except as $n = n'$, due to orthogonality of complex exponential function, thus leading to $\sum_n |X_n|^2$.

PSD…

3. Define PSD of a periodic function *x(t) as:*

$$PSD_x(f) = \sum_{n=-\infty}^{\infty} |X_n|^2 \delta(f - nf_0)$$

- Recall that a periodic function has <u>line spectra</u>.

$$X(f) = \sum_{n=-\infty}^{\infty} X_n \delta(f - f_0 n)$$

PSD…

- Then

$$
\begin{aligned}
P_x &= \int_{-\infty}^{\infty} PSD_x(f)\,df \\
&= \int_{-\infty}^{\infty} \left[\sum_{n=-\infty}^{\infty} |X_n|^2 \delta(f - nf_0) \right] df \\
&= \sum_{n=-\infty}^{\infty} |X_n|^2 \int_{-\infty}^{\infty} \delta(f - nf_0)\,df \\
&= \sum_{n=-\infty}^{\infty} |X_n|^2
\end{aligned}
$$

- Hence, $PSD_x(f)$ thus defined is, indeed, PSD.

PSD…

4. For <u>a non-periodic function *x(t)*</u>, we first truncate *x(t)* as $x_T(t)$ in $\left(-\frac{T}{2}, \frac{T}{2}\right)$ then find its FT: $X_T(f)$.

It can be shown that

$$PSD_x = \lim_{T\to\infty} \frac{1}{T} |X_T(f)|^2$$

Summary: ESD and PSD

- ESD: $\xi_x(f) = |X(f)|^2$
- PSD:

$$PSD_X(f) = \sum_{n=-\infty}^{\infty} |X_n|^2 \delta(f - nf_0), \quad x(t) \quad periodic$$

$$PSD_X(f) = \lim_{T\to\infty} \frac{1}{T} |X_T(f)|^2, \quad x(t) \quad non-periodic$$

$$X(f) = FT\{x(t)\}$$

$$X_n = FS \quad coefficient \quad of \quad x(t)$$

$$X_T(f) = FT\{x_T(t)\}$$

Autocorrelation

- Correlation function –

  Another approach to signal and systems.

- Autocorrelation –

  A measure of similarity (matching) of a signal with its delayed version.

Autocorrelation of an Energy Signal

- Definition

$$R_x(\tau) \overset{\Delta}{=} \int_{-\infty}^{\infty} x(t)x(t+\tau)dt, -\infty < \tau < \infty$$

- A measure of how closely the signal *x(t)* matches a copy of itself as the copy is shifted τ units in time
- The larger the more correlated.

Autocorrelation of an Energy Signal …

- Properties:

1. $R_x(\tau) = R_x(-\tau)$ symmetrical in τ about $\tau = 0$

2. $R_x(\tau) \leq R_x(0)\ \forall\tau$ maximum value of $R(\tau)$ occurs at $\tau = 0$

3. $R_x(\tau) \overset{FT}{\leftrightarrow} \xi_x(f) = ESD_x$ an important FT pair

4. $R_X(0) = \int_{-\infty}^{\infty} x^2(t)dt = E_X$ (energy)

Autocorrelation of a Power Signal

- Definition

$$R_X(\tau) \overset{\Delta}{=} \lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T} x(t)x(t+\tau)dt, -\infty < \tau < \infty$$

$$or = \langle x(t)x(t+\tau) \rangle$$

- For a periodic signal x(t) with period T_0

$$R_X(\tau) \overset{\Delta}{=} \frac{1}{T_0}\int_{-\frac{T_0}{2}}^{\frac{T_0}{2}} x(t)x(t+\tau)dt, -\infty < \tau < \infty$$

- Properties ($R_X(\tau)$ of a real-valued periodic signal *x(t)*):

1.	$R_x(\tau) = R_x(-\tau)$	even symmetry
2.	$R_x(\tau) \leq R_x(0), \ \forall \tau$	maximum value
3.	$R_x(\tau) \overset{FT}{\leftrightarrow} PSD_X(f)$	an important FT pair
4.	$R_X(0) = \frac{1}{T_0}\int_{-\frac{T_0}{2}}^{\frac{T_0}{2}} x^2(t)dt$	average power

Random Signals

- Random experiment: E
- Outcome of r.e.: ξ
- Sample space: S
- Definition of probability
- Axioms of probability
- Random event: A
- Random variable (r.v.): X(A)

  a function of A, a real number,
  (numerical attribute)

Random Signals
CDF

- CDF Cumulative Distribution Function

$$F_X(x) = P(X \leq x)$$

- Properties:

1. $0 \leq F_X(x) \leq 1$
2. $F_X(x_1) \leq F_X(x_2), \quad if \quad x_1 \leq x_2$
3. $F_X(-\infty) = 0$
4. $F_X(+\infty) = 1$
5. $F_X(x)$ continuous from the right

Random Signals
PDF

- PDF

$$f_X(x) = \frac{dF_X(x)}{dx}$$

$$\begin{aligned} P(x_1 \le X \le x_2) &= P(X \le x_2) - P(X \le x_1) \\ &= F_X(x_2) - F_X(x_1) \\ &= \int_{x_1}^{x_2} f_X(x)\,dx \end{aligned}$$

- Properties

1. $f_X(x) \ge 0$

2. $\int_{-\infty}^{\infty} f_X(x)dx = F_X(\infty) - F_X(-\infty) = 1$

- For discrete r.v.'s, instead of f_X *(f)*, we often use $P(X = x_i)$

  → probability mass function (PMF)

- Strictly speaking

$$\begin{cases} f_X(x) = \sum_i P(X = x_i)\delta(x - x_i) & \text{pdf of discrete r.v.} \\ \qquad \forall i; x_i < x & \\ F_X(x) = \sum_i P(X = x_i)u(x - x_i) & \text{cdf of discrete r.v.} \end{cases}$$

Ensemble Average (statistical average)

- Numerical attributes of r.v.
  - Mean value, m_X (expected value)

$$m_X = E(X) = \int_{-\infty}^{\infty} x f_X(x)dx$$

  - n^{th} moment

$$E(X^n) = \int_{-\infty}^{\infty} x^n f_X(x)dx$$

n = 1, mean value of X, mean
n = 2, mean-square value of X
n = 3,
n = 4,

} very important (this and next slides)

Ensemble Average (statistical average)…

– Central moment

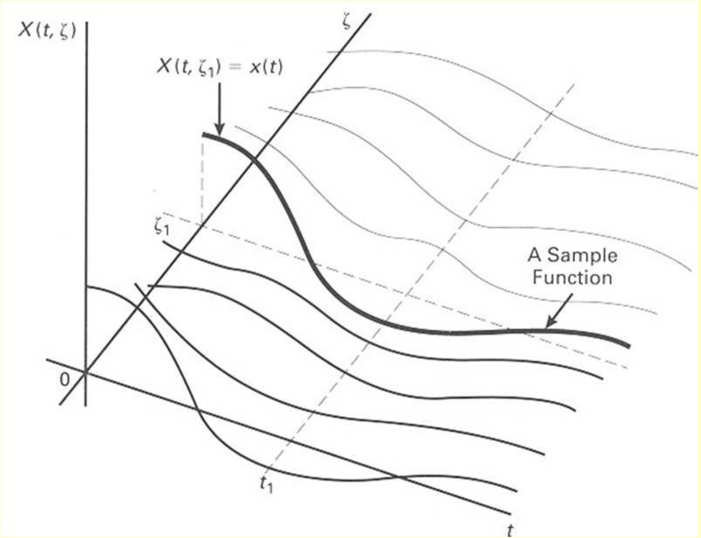
$$E\left[(X - m_X)^n\right] = \int_{-\infty}^{\infty} (x - m_X)^n f_X(x)dx$$

1st central moment: 0
2nd central moment: $E\left[(X - m_X)^2\right] = \mathrm{var}(X) = \sigma_X^2$
Important formula: $\sigma_X^2 = E[X^2] - m_X^2$ variance
3rd central moment: skewness of pdf
4th central moment: kurtosis of pdf

Random Process (r.p.)

- *X (A, t)* A: random event
  t: time
  - a larger set
  - a collection of r.v.’s
  - or, a collection of random sample function
  - often simplified as *X (t)*

Figure 8.1-1 from <Probability, Random Processes, and Estimation Theory for Engineers> by Stark & Woods 1994



- For a specified event A_j, X (A_j, t) = X_j (t), a sample function
- For a specified time t_k, X (A, t_k) = X_k, a r.v.
- For specific A_j, t_k, X (A_j, t_k) = a real value

Statistical Average of a r.p.

- A r.p. should be completely characterized by pdf of all r.v.'s. This is difficult.
- Hence, numerical characterization used often.
- A partial description:
  - **mean function** (1^{st} order)
  - **autocorrelation function** (2^{nd} order)

$$E[X(t_k)] = \int_{-\infty}^{\infty} x f_{X_k}(x)dx = m_X(t_k)$$

$$R_X(t_1, t_2) = E[X(t_1) \cdot X(t_2)]$$

Stationarity

- <u>S.S.S. (Strict Sense Stationary)</u>

  A r.p. X(t) is SSS if none of its statistics are affected by a shift in the time origin.
- <u>W.S.S.(Wide Sense Stationary)</u>

  A r.p. X(t) is WSS if its mean and autocorrelation function do not vary with a shift in the time origin.

Stationarity …

- For a W.S.S. random process,

  $E[X(t)] = m_X$ a constant

  $R_X(t_1, t_2) = R_X(t_1 - t_2)$ only a function of $t_1 - t_2$
- Relation between SSS and WSS

$$SSS \quad \underset{\textit{not} \quad \textit{nessarily}}{\overset{\rightarrow}{\not\leftarrow}} \quad WSS$$

Autocorrelation of a WSS r.p.

- Let τ = $t_1 - t_2$

$$R_X(\tau) = E[X(t)X(t+\tau)], -\infty < \tau < \infty$$

- $R_x(\tau)$ gives us an idea of the frequency response
- If $R_x(\tau)$ changes slowly as τ changes, then X (t) has more low frequency components; otherwise, it has more high frequency comp.
- Four similar properties of $R_x(\tau)$ to that of autocorrelation of an energy (power) signal.

Time Average & Ergodicity

- To compute m_X and $R_x(\tau)$ needs to have $f_X(x)$ for all *x*, sometimes not possible,
  ⇒ prefer the time average
- Time average over a single sample function of the r.p.:

$$\left.\begin{array}{ll} \langle X(t)\rangle & \lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T}X(t)dt \\ \langle X(t)X(t+\tau)\rangle & \lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T}X(t)X(t+\tau)dt \end{array}\right\} \textit{always} \quad \textit{possible}$$

- An ergodic r.p.: $m_X(t) = \langle X(t) \rangle$

$$R_X(\tau) = \langle X(t)X(t+\tau) \rangle$$

$$\mathit{ergodicity} \;\underset{\mathit{not\quad nessarily}}{\overset{\rightarrow}{\not\leftarrow}}\; \mathit{SSS}$$

- A r.p. is ergodic in the mean, if

$$m_X = \langle X(t) \rangle$$

- A r.p. is ergodic in the autocorrelation function if

$$R_X(\tau) = \langle X(t)X(t+\tau) \rangle$$

- *A reasonable assumption* in the analysis of most communication signals (in the absence of transient effects) is that random waveforms are ergodic in the mean and autocorrelation function
- For ergodic r.p., time averages = statistical (ensemble) averages

- some observations:
  - m_x = dc levels of the signal
  - m_x^2 = normalized power of dc component
  - E $[X^2]$ = total average normalized power
  - $\sqrt{E[X^2]}$ = root-mean-square (rms) value of X
  - σ_x^2 = average normalized power in ac component
  - If $m_X = 0$, σ_x^2 = mean-square value of X or total power
  - σ_x = rms of ac component
  - If $m_X = 0$, σ_x = rms of the signal

PSD of a Random Process

- A r.p. X(t) can generally be considered as a power signal.
- *PSD$_X$ (f):*

$$PSD = \lim_{T\to\infty} \frac{1}{T} |X_T(f)|^2$$

$$x(t) \to x_T(t) \qquad as \qquad t \in \left(-\frac{T}{2}, \frac{T}{2}\right) \leftrightarrow X_T(f)$$

- Useful in communication systems:
  Indicates the distribution of a signal's power in frequency domain.

PSD of a Random Process…

- Properties:
  1. $PSD_X(f) \geq 0$ nonnegative real-valued
  2. $PSD_X(f) = PSD_X(-f)$ if X(t) is real-valued
  3. $R_X(\tau) \overset{FT}{\leftrightarrow} PSD_X(f)$
  4. $P_X = \int_{-\infty}^{\infty} PSD_X(f)df$

Noise in Communication Systems

- Noise: Unwanted electrical signals
  always present in electrical system

<u>Man-made Noise</u>: switching transients,
spark-plug ignition

<u>Natural Noise</u>: thermal (always exists)
Gaussian noise (central
limit theorem (slide 54))

Normalized Gaussian pdf

- $$\left.\begin{matrix} m = 0 \\ \sigma^2 = 1 \end{matrix}\right\} \Rightarrow$$

$$f(n) = \frac{1}{\sigma\sqrt{2\pi}} \exp\left[-\frac{1}{2}\left(\frac{n}{\sigma}\right)^2\right]$$

Gaussian Noise

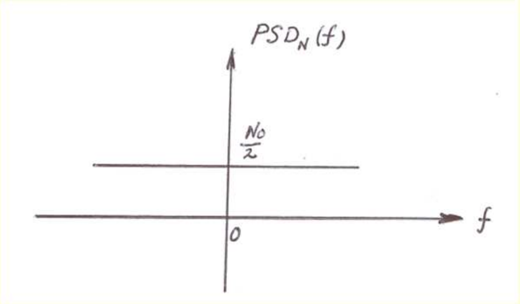
- A random signal $z = a + n$
  a: deterministic component of the signal
  n: additive random noise

$$f(z) = \frac{1}{\sigma\sqrt{2\pi}} \exp\left[-\frac{1}{2}\left(\frac{z-a}{\sigma}\right)^2\right]$$

- Gaussian distribution is extremely important in practice due to the central limit theorem
- The theorem: The probability distribution of the sum of j statistically independent r.v.'s approaches the Gaussian distribution as $j \rightarrow \infty$, no matter what the individual distribution functions may be.

White Noise

- Def.



- "White" comes from the fact that white light contains equal amounts of all frequencies within the visible band of electromagnetic radiation

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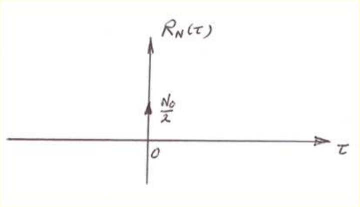
- Autocorrelation function

$$R_n(\tau) = FT^{-1}\{PSD_X(f)\} = FT^{-1}\left\{\frac{N_0}{2}\right\} = \frac{N_0}{2}\delta(\tau)$$



n(t) is totally decorrelated from its time-shifted version for any τ > 0

- Meaning: Any two different samples of a white noise are uncorrelated no matter how close together in time they are taken.

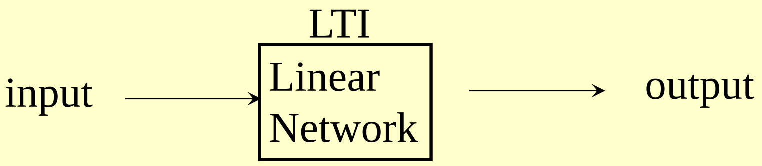
- Average power $P_n = \int_{-\infty}^{\infty} \frac{N_0}{2} df = \infty$

- In practice, as long as the bandwidth of the noise is appreciably larger than that of the system, the noise can be considered to have an infinite bandwidth.

- Thermal noise:

  Additive white Gaussian Noise (AWGN)
  Very important in both theory and practice.

Signal Transmission Through Linear Systems



- x(t), h(t), y(t) --- time domain
- X(f), H(f), Y(f) --- frequency domain
- h(t): unit impulse response of the LTI system
- H(f): transfer function

- **h(t)**: **unit impulse response**
  - $h(t) = y(t)$ when $x(t) = \delta(t)$
  - $y(t) = x(t) * h(t) = \int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau = \int_{-\infty}^{\infty} x(t-\tau)h(\tau)d\tau$
- **Causality**
  - A system is causal if there is no output prior to the time, $t = 0$, when the input is applied.
  - A system is causal if $h(t) = 0, \quad \forall t < 0$

$$\Rightarrow y(t) = \int_{0}^{\infty} x(\tau)h(t-\tau)d\tau$$

- **H(f): Frequency transfer function**.

$$\begin{cases} \text{transfer function.} \\ \text{frequency response} \end{cases}$$

- $H(f) = FT\{h(t)\}$
- $Y(f) = FT\{y(t)\} = FT\{x(t) * h(t)\} = X(f) \cdot H(f)$
- $H(f) = \dfrac{Y(f)}{X(f)}$

Frequency Response

- *H(f)* : Complex in general

$$H(f) = |H(f)| e^{j\theta(f)}$$

- Amplitude frequency response: $|H(f)|$
- Phase frequency response: $\theta(f)$

- The transfer function of a LTI system can be measured by using a sinusoidal testing signal (that is swept over the frequency of interest) since the spectrum of sinusoid is a line at the testing frequency.
- For sinusoidal input, *i.e.*, $x(t) = A\cos[2\pi f_0 t + \phi)]$

$$y(t) = \underbrace{A|H(f_0)|}_{magnitude} \cos[2\pi \underset{\substack{\downarrow \\ same \quad frequency}}{f_0} t + \underbrace{\phi + \theta(f_0)}_{\substack{\downarrow \\ new \quad phase}})]$$

Random Process & Linear Systems

- If $x(t)$: a r.p.
  $h(t)$: LTI system
  then $y(t)$: output, another r.p.
  i.e., every sample function of the input r.p.
  → a corresponding sample function

- $PSD_x(f)$: PSD of x(t)
  $PSD_y(f)$: PSD of y(t)

$$\boldsymbol{PSD_y(f) = PSD_x(f)\cdot|H(f)|^2}$$

- The relation between the power spectral density (PSD) at the input, $PSD_x(f)$, and the output, $PSD_y(f)$

$$PSD_y(f) = |H(f)|^2\ PSD_x(f)$$

- The power transfer function of the LTI system

$$G_h(f) = \frac{PSD_y(f)}{PSD_x(f)} = |H(f)|^2$$

$$\because PSD_y(f) = \lim_{T\to\infty}\frac{1}{T}|Y_T(f)|^2 \qquad Y(f) = X(f)H(f)$$

$$PSD(f)_x = \lim_{T\to\infty}\frac{1}{T}|X_T(f)|^2$$

- If *x(t)* is a Gaussian r.p., *h(t)* is LTI, then *y(t)* is also Gaussian
- If the input to a LTI system is periodic with spectrum given by

$$X(f) = \sum_{n=-\infty}^{\infty} X_n \delta(f - n \cdot f_0)$$

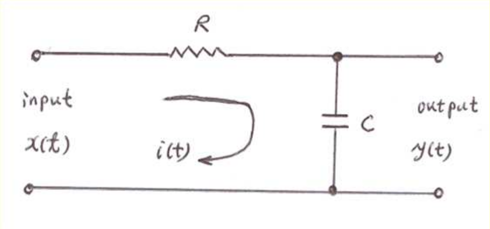
line spectrum

where $\{X_n\}$ is the complex exponential FS coefficients of the input signal, then the output signal's spectrum is

$$Y(f) = \sum_{n=-\infty}^{\infty} X_n H(nf_0)\delta(f - n \cdot f_0)$$

RC Low-Pass Filter

- Example 3-1



x(t) = R i(t) + y(t)

$$\because i(t) = C\frac{dy(t)}{dt} \Rightarrow RC\frac{dy(t)}{dt} + y(t) = x(t)$$

From the FT property of the differentiation,

$$\Rightarrow RC(j2\pi f)Y(f) + Y(f) = X(f)$$

$$H(f) = \frac{Y(f)}{X(f)} = \frac{1}{1 + j(2\pi RC)f}$$

$$or \quad H(f) = \frac{\frac{1}{j\omega C}}{R + \frac{1}{j\omega C}} = \frac{1}{1 + j\omega RC} = \frac{1}{1 + j2\pi fRC}$$

$$h(t) = \begin{cases} \frac{1}{\tau_0} e^{-\frac{t}{\tau_0}} & t \ge 0 \\ 0 & t < 0 \end{cases}$$

with τ_0 = RC: time constant

according to Table 3.1

$$G_h(f) = |H(f)|^2 = \frac{1}{1 + \left(\frac{f}{f_0}\right)^2}, \qquad f_0 = \frac{1}{2\pi RC}$$

power transfer function

Distortionless Transmission

- What is an ideal transmission line?
- In time domain
  - Some time delay is allowed ($y(t)$ vs. $x(t)$)
  - A scale change in magnitude is allowed

$\Rightarrow y(t) = K\,x(t-t_0)$

K: scale change

t_0: time delay

- In frequency domain

$$Y(f) = K \cdot X(f) e^{-j2\pi f t_0}$$

$$i.e., \quad H(f) = K \cdot e^{-j2\pi f t_0}$$

- |H(f)| = K constant magnitude change $\forall f$
- $\theta(f) = \angle H(f) = -2\pi f t_0$

$$\Rightarrow \quad t_0 = \frac{-\theta(f)}{2\pi f}$$

t_0 needs to be fixed.

→ Phase shift must be proportional to frequency in order for the time delay of all components to be identical, i.e.,

$$phase\_delay\_\theta \propto f$$

“Equalization”: phase or amplitude correction network

Ideal Filters

- One cannot build the ideal network described above since it implies an infinite bandwidth capability (Sklar’s , page 33).
- An approximation to the ideal infinite-bandwidth network is to use a truncating network that passes all freq. components between f_l and f_u without distortion, where f_l and f_u are the lower and upper cutoff frequency, respectively.
- Ideal BPF
  LPF
  HPF

Ideal Filters

Fig. 1.10 from <Digital Communications> by Sklar,1988

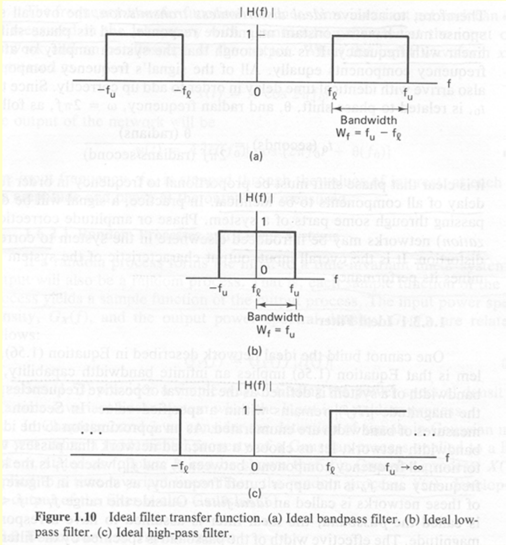


Figure 1.10 Ideal filter transfer function. (a) Ideal bandpass filter. (b) Ideal low-pass filter. (c) Ideal high-pass filter.

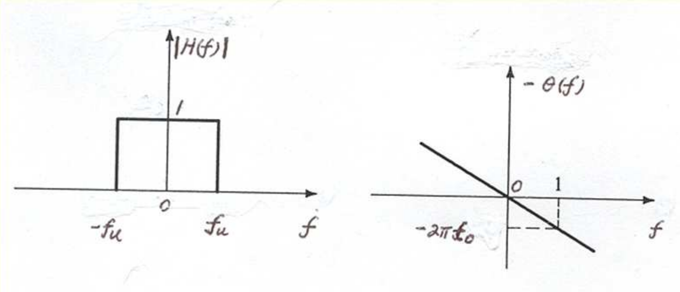
- Take a look at ILPF

Why more emphasize LPF?

$$H(f) = |H(f)|e^{-j\theta(f)}$$

$$|H(f)| = \begin{cases} 1 & for\,|f| < f_u \\ 0 & for\,|f| \geq f_u \end{cases}$$

$$e^{-j\theta(f)} = e^{-j2\pi f t_0}$$



$$h(t) = FT^{-1}\{H(f)\} = \int_{-\infty}^{\infty} H(f) e^{-j2\pi ft} df$$

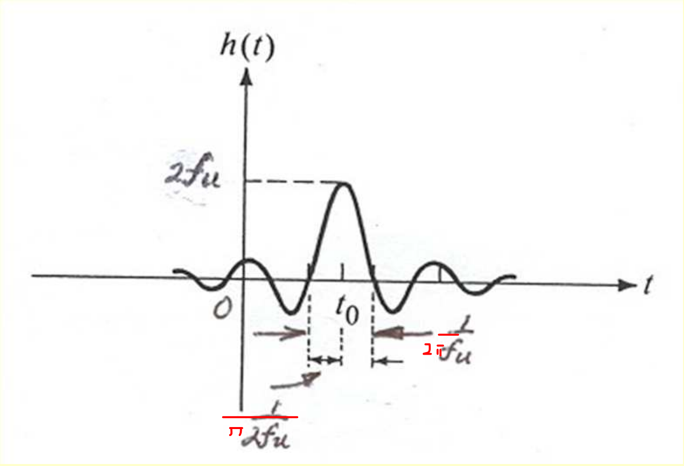
$$= \int_{-f_u}^{f_u} e^{-j2\pi ft_0} e^{j2\pi ft} df$$

$$= \int_{-f_u}^{f_u} e^{j2\pi f(t-t_0)} df$$

$$= 2 f_u \frac{Sin[2\pi f_u (t-t_0)]}{2\pi f_u (t-t_0)}$$

$$= 2 f_u Sinc\left[2\pi f_u (t-t_0)\right]$$

Unit Impulse Response of ILPF



Effect of an ILPF on White Noise

- Example 1.2.

$$PSD_n(f) = \frac{N_0}{2} \quad PSD_Y(f) = ? \quad R_Y(\tau) = ?$$

- Solution:

$$PSD_Y(f) = PSD_n(f)\,|H(f)|^2$$

$$= \begin{cases} \frac{N_0}{2} & for\,|f| < f_u \\ 0 & otherwise \end{cases}$$

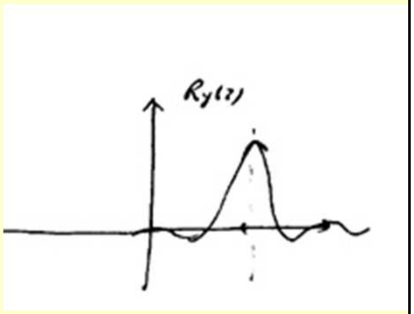
$$R_Y(\tau) = FT^{-1}\{PSD_Y(f)\}$$

$$= \underbrace{2 \cdot f_u \cdot \frac{N_0}{2}}_{area\_of\_PSD_Y} sinc[\underbrace{2f_u}_{width\_of\_PSD_Y(f),} \tau] = N_0 f_u \, sinc[2\pi f_u \tau]$$

$\frac{1}{2f_u}$: 1st zero crossing

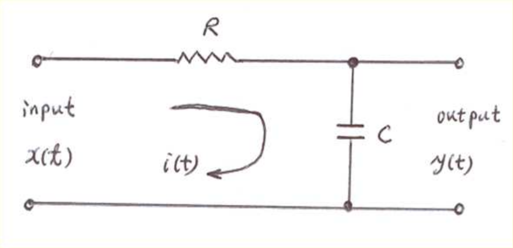
After LPF, is it white noise or not?

Not a white noise anymore



Realizable Filters

- RC LPF:

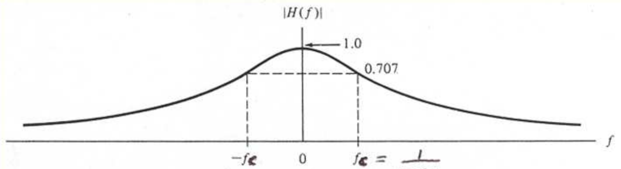


- RC filter (frequency analysis of sinusoidal circuits)

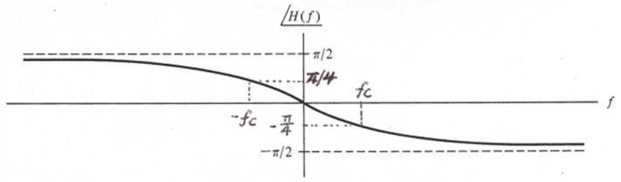
$$\frac{V_{out}}{V_{input}} = \frac{1 \big/ j2\pi fc}{R + 1 \big/ j2\pi fc} = \frac{1}{1 + j2\pi fRc}$$

$$|H(f)| = \frac{1}{\sqrt{1+(2\pi fRc)^2}}$$

$$\theta(f) = -\tan^{-1}(2\pi fRc)$$



(a) Amplitude response



(b) Phase response

Amplitude and phase responses for a low-pass *RC* filter.

- Consider R = 1, i.e., the normalized case.

$$P = \frac{V^2}{R} = V^2$$

$$\frac{V_2}{V_1} = \frac{\sqrt{2}}{2} \Rightarrow \frac{V_2{}^2}{V_1{}^2} = \frac{1}{2}, \frac{P_2}{P_1} = \frac{1}{2}$$

$$\frac{V_2}{V_1} = 0.707 \Leftrightarrow \frac{P_2}{P_1} = \frac{1}{2} \qquad half - power$$

- No. of dB $= 10\log_{10}\left(\frac{P_2}{P_1}\right) = -10\log_{10} 2 = -10 \cdot 0.3010 \approx -3dB$

- Hence, *half-power* ⇔ *-3dB*

Effect of an RC filter on white noise

- Example 1.3

$$G_n = \frac{N_0}{2}$$

$$G_y(f) = G_n(f) \cdot |H(f)|^2 = \frac{N_0}{2} \cdot \frac{1}{1 + (2\pi f Rc)^2}$$

$$R_y(\tau) = F^{-1}\{G_y(f)\} = \underbrace{\frac{N_0}{4Rc} exp\left(-\frac{|\tau|}{Rc}\right)}_{exponential\_function}$$

When input is white noise, output of the RC filter: Not white noise anymore

Several Useful Realizable Filters

- Butterworth filter (most flat one in passband)
- Chebychev filter
  Ripple in passband
  smaller variation in stopband
- For pass-band and stop-band refer to the figure on slide 82

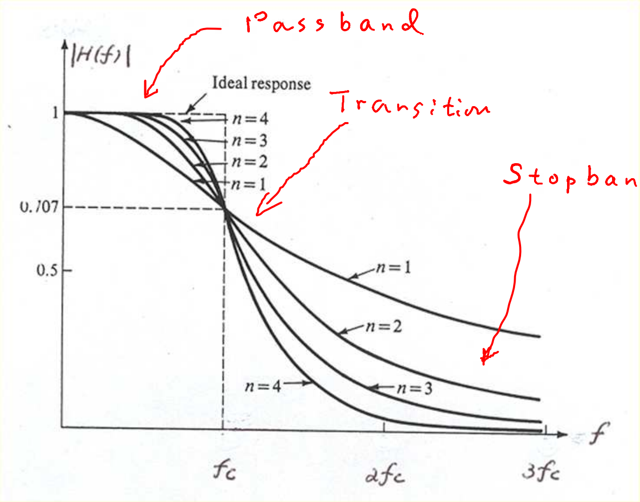
Butterworth Low Pass Filter

$$|H_n(f)| = \frac{1}{\sqrt{1+\left(\frac{f}{f_u}\right)^{2n}}}, n \geq 1$$

f_u, or f_c *called corner frequency*

$n \to \infty, H_n(f) \to ILPF$

Butterworth Amplitude Response



Signals, Circuits & Spectra

- Input signal *x(t)*, its spectrum |*X(f)*|
  rect(t) |*sinc(f)*| e.g.
- Circuit, RC circuit, |*H(f)*|, *θ(f)*
- Output *y(t)*, |*Y(f)*|
- Case 1: Output bandwidth is constrained by input signal bandwidth,
  i.e., H(f) is wideband.
- Case 2: Output bandwidth is constrained by filter bandwidth,
  i.e., H(f) is narrowband.

Fig. 1.14, from <Digital Communications> by Sklar,1988

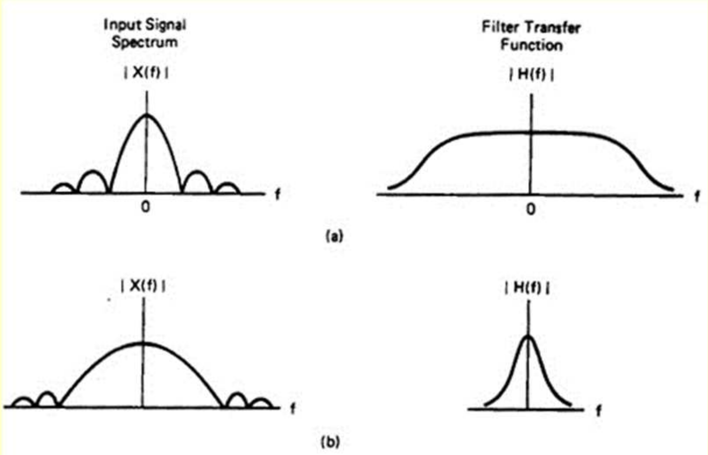


Figure 1.14 Spectral characteristics of the input signal and the circuit contribute to the spectral characteristics of the output signal. (a) Case 1: Output bandwidth is constrained by input signal bandwidth. (b) Case 2: Output bandwidth is constrained by filter bandwidth.

Bandwidth of Digital Data

- Baseband vs. Bandpass
  - Double sideband (DSB) modulation
    x(t) signal (low-pass or baseband signal)
    |X(f)| spectrum, $0 - f_m$: baseband bandwidth
  - DSB modulated signal

$$x_c(t) = x(t)\cos(2\pi f_c t),\ f_c \ge f_m$$

baseband signal $x(t) \rightarrow \otimes \rightarrow x_c(t)$ DSB modulated signal

$$\uparrow$$

$\cos(2\pi f_c t)$ local oscillator (LO) carrier

$$X_c(f) = \frac{1}{2}\left[X(f - f_c) + X(f + f_c)\right]$$

$$\because\ F\{\cos(2\pi f_c t)\} = \frac{1}{2}\left[\delta(f - f_c) + \delta(f + f_c)\right]$$

and convolution of a normal factor with a δ function.