# Fourier Techniques
# Chapters 2 & 3, Part I

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Text used for the course: <Modern Digital and Analog Communication Systems>, 4th Edition, Lathi and Dong, Oxford



## Fourier Technique Review:
## Complex Exponential Fourier Series

- Complex Exponential Fourier Series
  Consider a signal *x (t)*, $t_0 \le t \le t_0 + T_0$
- *x (t)* satisfies Dirichlet conditions:
  *a)* *x (t)* has only a finite number of maxima & minima.
  b) The number of discontinuities must be finite.
  c) The discontinuities must be bounded.

## Complex Exponential Fourier Series…

- Then,

$$x(t) = \sum_{n=-\infty}^{\infty} X_n e^{jn\omega_0 t} \qquad t_0 \le t \le t_0 + T_0$$

$$X_n = \frac{1}{T_0} \int_{t_0}^{t_0+T} x(t) e^{-jn\omega_0 t} dt$$



## Complex Exponential Fourier Series…

- This complex exponential Fourier series represent x(t) exactly, except at a point of jump discontinuity where it converges to the arithmetic mean of the left- and right-hand limits
- Dirichlet conditions are sufficient conditions, not necessary conditions.
- Outside the interval, nothing is guaranteed
- All of the periodic functions in practice obey Dirichlet conditions

## Sufficient Condition and Necessary Condition

- Condition and Statement
- Sufficient condition (s.c.): If s.c. is satisfied, it is guaranteed that the statement holds. (if)
  - If s.c. is not satisfied, it is not clear if the statement holds.
- Necessary condition (n.c.): If the statement holds, it is necessary that n.c. holds. (only if)
  - If n.c. holds, it is not clear if the statement holds.
- Some condition is necessary and sufficient (if and only if (iff)).



## Complex Exponential Fourier Series…

- The above complex exponential Fourier series (slide 3) can be viewed as a complete orthonormal series expansion

$$\phi_n(t) = e^{jn\omega_0 t} \qquad n = 0, \pm 1, \pm 2, \ldots$$

$\Rightarrow$ The complex exponential Fourier series are unique:

- If one finds a F.S. for x(t) then there is no other F.S. expansion for x(t)

## Orthogonality

- Definition
- A set of M signals

$$y_1(t), y_2(t), \ldots, y_M(t)$$

is said to be orthogonal, if

$$\int_0^{T_b} y_i(t) y_j(t) dt = \begin{cases} c & i = j \\ 0 & i \neq j \end{cases}$$



## Example: **Sinusoidal function** – *sin(k·ωt)*

## An orthogonal signal

- One example of M orthogonal signals

$$y_k(t) = \begin{cases} Sin\dfrac{2\pi kt}{T_b} & 0 < t < T_b \quad k = 1,2,\ldots,M \\ 0 & otherwise \end{cases}$$

In the set, M different frequencies are:

$$k\frac{1}{T_b} : \frac{1}{T_b}, \frac{2}{T_b}, \ldots, \frac{M}{T_b}$$

Note: Integration of $Sin^2(x) \neq 0$ in $2\pi$, integration of $Sin(x)Sin(2x)=0$ in $2\pi$.

## Complex Exponential Fourier Series Coefficients: Time average

- $$X_n = \frac{1}{T_0} \int_{t_0}^{t_0+T_0} x(t) e^{-jn\omega_0 t} dt$$
- Define the time average as

$$\langle \upsilon(t) \rangle = \begin{cases} \lim_{T \to \infty} \frac{1}{2T} \int_{-T}^{T} \upsilon(t) dt & \text{If } \upsilon(t) \text{ not periodic} \\ \frac{1}{T_0} \int_{T_0} \upsilon(t) dt & \text{If } \upsilon(t) \text{ periodic} \end{cases}$$



## Complex Exponential Fourier Series Coefficients: Time average …

- $$X_0 = \frac{1}{T_0} \int_{t_0}^{t_0+T_0} x(t) dt = \langle x(t) \rangle$$

= the time average of *x(t)* , or dc value of *x(t)*

$$X_n = \frac{1}{T_0} \int_{t_0}^{t_0+T_0} x(t) \cos n\omega_0 t dt - j \frac{1}{T_0} \int_{t_0}^{t_0+T_0} x(t) \sin n\omega_0 t dt$$

$$= \langle x(t) \cos n\omega_0 t \rangle - j \langle x(t) \sin n\omega_0 t \rangle$$

## Complex Exponential Fourier Series Coefficients: Symmetry

- If x(t) is real, then

$$X_n^* = X_{-n}$$

According to definition
Also can be seen from above

$$\begin{Bmatrix} X_n = |X_n| e^{j\angle X_n} \\ X_{-n} = |X_{-n}| e^{j\angle X_{-n}} \end{Bmatrix} \Rightarrow \begin{Bmatrix} |X_n| = |X_{-n}| \\ \angle X_n = -\angle X_{-n} \end{Bmatrix}$$

$\Rightarrow$ Magnitude: Even symmetry
Phase angle : Odd symmetry



## Complex Exponential Fourier series Coefficients: Symmetry …

- If x(t) is real and even, i.e., *x(t)* = *x(-t)*

$$X_n = \langle x(t) \cos n\omega_0 t \rangle$$

$$\because \left\langle \underbrace{x(t) \sin n\omega_0 t} \right\rangle = 0$$

(An odd function of *t* taking integration in an even interval with respect to *t=0* axis.)

$X_n$ is real

$X_n$ is even w.r.t. *n*

## Complex Exponential Fourier series Coefficients: Symmetry …

- If x(t) is real and odd, i.e., $x(t) = -x(-t)$

$$X_n = -j\langle x(t)\sin n\omega_0 t\rangle \neq 0 \quad \text{Imaginary, odd function of } n$$

$$\because \left\langle \underbrace{x(t)\cos n\omega_0 t}_{odd} \right\rangle = 0$$

(An odd function of *t* taking integration in an even interval with respect to *t=0* axis.)



## Fourier Technique Review: **Trigonometric Form of Fourier Series**

- **Compact form** of trigonometric FS (also referred to as **Polar form** FS)

$$x(t) = X_0 + \sum_{n=1}^{\infty} 2|X_n|\cos(n\omega_0 t + \angle X_n)$$

$x(t)$ is real
$X_n$ is Fourier Coefficient

## Trigonometric Form of Fourier Series (continue)

- **Quadrature** form of FS

$$x(t) = \frac{A_0}{2} + \sum_{n=1}^{\infty} A_n \cos n\omega_0 t + \sum_{n=1}^{\infty} B_n \sin n\omega_0 t$$

$$A_n = \frac{2}{T_0} \int_{t_0}^{t_0+T_0} x(t)\cos(n\omega_0 t)dt, \qquad n = 0,1,\cdots$$

$$B_n = \frac{2}{T_0} \int_{t_0}^{t_0+T_0} x(t)\sin(n\omega_0 t)dt, \qquad n = 1,2,\cdots$$

$$\left( X_0 = \frac{1}{T_0} \int_{t_0}^{t_0+T_0} x(t)dt = \frac{A_0}{2} \right)$$



## Trigonometric Form of Fourier Series

- In either the trigonometric or the complex exponential form of the F.S.
  - $X_0$ : the average of *x(t)* (or dc component of *x(t)*)
  - $X_1$ : the fundamental component
  - $X_2$ : the second harmonic component
  - $X_3$ : the third harmonic component

    ⋮
- Meaning

## Trigonometric Form of Fourier Series (derivation from complex exponential form)

$$x(t) = \sum_{n=-\infty}^{\infty} X_n e^{jn\omega_0 t}$$

$$= \sum_{n=-\infty}^{\infty} |X_n| e^{j\angle X_n}.(\cos n\omega_0 t + j\sin n\omega_0 t) \qquad (\because \quad Euler \quad formula)$$

$$= \sum_{n=1}^{\infty} [|X_n|\cos\angle X_n + j|X_n|\sin\angle X_n][\cos n\omega_0 t + j\sin n\omega_0 t]$$

$$+ \sum_{n=-1}^{-\infty} [|X_n|\cos\angle X_n + j|X_n|\sin\angle X_n][\cos n\omega_0 t + j\sin n\omega_0 t] + X_0$$



If $x(t)$ is real, $\Rightarrow |X_n|$ is even, $\angle X_n$ is odd, w.r.t. n

$$\Rightarrow x(t) = X_0 + \sum_{n=1}^{\infty} [|X_n|\cos\angle X_n \cos n\omega_0 t - |X_n|\sin\angle X_n \sin n\omega_0 t$$

$$+ j|X_n|\sin\angle X_n \cos n\omega_0 t + j|X_n|\cos\angle X_n \sin n\omega_0 t]$$

$$+ \sum_{n=1}^{\infty} [|X_n|\cos\angle X_n \cos n\omega_0 t - |X_n|\sin\angle X_n \sin n\omega_0 t$$

$$- j|X_n|\sin\angle X_n \cos n\omega_0 t - j|X_n|\cos\angle X_n \sin n\omega_0 t]$$

$$= X_0 + \sum_{n=1}^{\infty} 2|X_n|[\cos\angle X_n \cos n\omega_0 t - \sin\angle X_n \sin n\omega_0 t]$$

$$= X_0 + \sum_{n=1}^{\infty} 2|X_n|\cos(n\omega_0 t + \angle X_n)$$

## A formula in derivation

$$\sum_{n=1}^{\infty} 2|X_n|\cos(n\omega_0 t + \angle X_n)$$
$$= \sum_{n=1}^{\infty} 2|X_n| \frac{e^{jn\omega_0 t + j\angle X_n} + e^{-jn\omega_0 t - j\angle X_n}}{2}$$
$$= \sum_{n=1}^{\infty} |X_n| e^{jn\omega_0 t} e^{j\angle X_n} + \sum_{n=1}^{\infty} |X_n| e^{-jn\omega_0 t} e^{-j\angle X_n}$$
$$= \sum_{n=1}^{\infty} X_n e^{jn\omega_0 t} + \sum_{n=1}^{\infty} X_{-n} e^{-jn\omega_0 t}$$
$$= \sum_{n=1}^{\infty} X_n e^{jn\omega_0 t} + \sum_{n=-\infty}^{-1} X_n e^{jn\omega_0 t}$$



## Definition of Sinc Function

$$sinc(x) = \frac{\sin(\ x)}{x}$$

- Also, called interpolating function.
- Even function of *x*.
- Equal to *0* as $x \neq 0$ & $\sin x = 0$
  ($\sin x = 0$ for $x = n\pi$ as *n* is an integer).
- Note that *sinc (0) =1* due to L'Hopital's rule.
- *sinc(x)* exhibits sinusoidal oscillations with amplitude decreasing continuously as *1/x*.

## Figure of *sinc(x)*

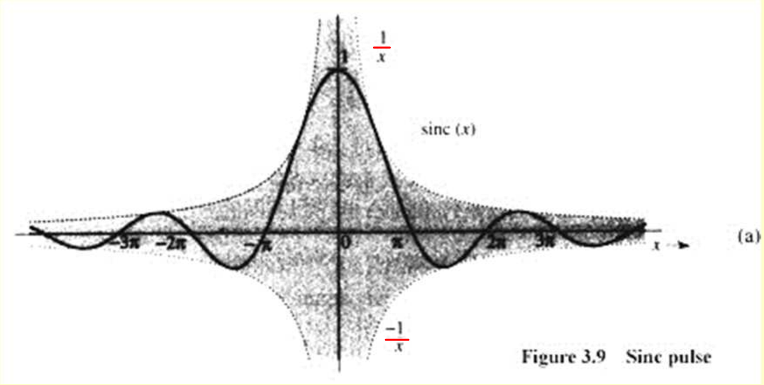



Figure 3.9 Sinc pulse



## Example:
## Rectangle function and its FS

- General form:

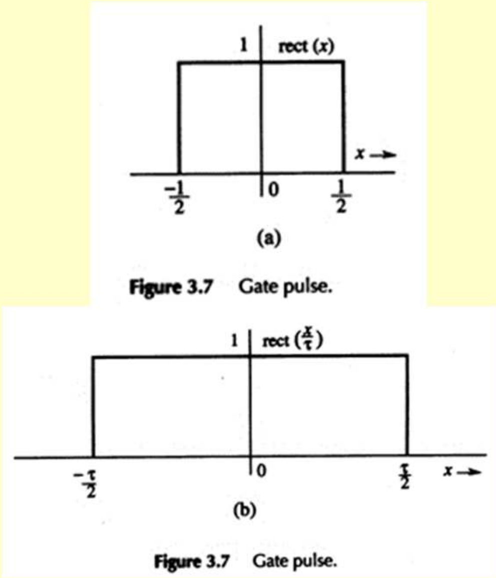
$$\prod(t) = \begin{cases} 1 & |t| < \frac{1}{2} \\ 0 & otherwise \end{cases}$$

$$\prod\left(\frac{t}{\tau}\right) = \begin{cases} 1 & |t| < \frac{\tau}{2} \\ 0 & otherwise \end{cases}$$

- Most general form: A, $t_0$, $\tau$

$$A\prod\left(\frac{t-t_0}{\tau}\right)$$

# Figure of rect(t)






$$X_n = \frac{1}{1}\int_{-1/2}^{1/2} \prod(t)\cdot e^{-jn\omega_0 t}\,dt$$

$$= \int_{-1/2}^{1/2} \left(\cos(-n\omega_0 t) + j\sin(-n\omega_0 t)\right)dt$$

$$= 2\int_{0}^{1/2} \cos(n\omega_0 t)\,dt = 2\left.\frac{\sin n\omega_0 t}{n\omega_0}\right|_0^{1/2}$$

$$= 2\left.\frac{\sin 2\pi n t}{2\pi n}\right|_0^{1/2} = \frac{Sin(\pi n)}{\pi n} = \sin c(\pi n)$$

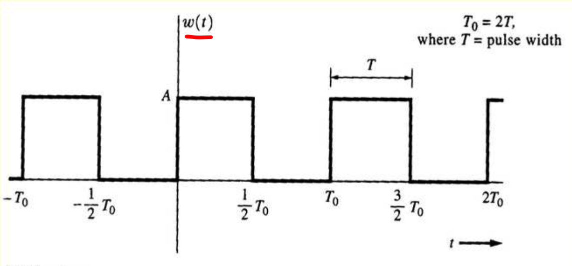
## Rectangle (window) function and its FS

- When $n=0, X_0=1$.
- When $n \neq 0$ , $X_n = 0$ .
- These make sense for *rect(t)*, don't they?



## Spectrum (FS) of A Rectangular Pulse Train

- The pulse train is a signal of great interest in digital communications
  – A periodic sequential of rectangular pulses



(a) Waveform

## Spectrum of the Rectangle Pulse Train …

- Its complex Fourier series coefficient

$$X_n = \frac{1}{T_0}\int_0^{T_0/2} Ae^{-jn\omega_0 t}dt = j\frac{AT_0}{2\pi n}(e^{-jn\pi} - 1)$$

$$X_n = \frac{A}{T_0}V(nf_0) = \frac{A}{2}e^{-jn\pi/2}\frac{\sin(n\pi/2)}{n\pi/2}$$

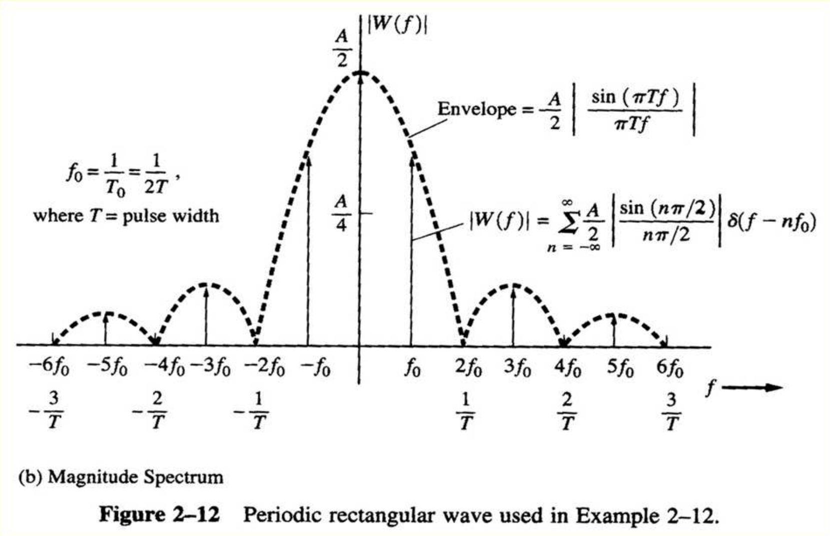
where *V(f)* is the Fourier Transform of a period of *w(t)* [which is a rectangle].



$$X_n = \begin{cases} \frac{A}{2}, & n = 0 \\ -j\frac{A}{n\pi}, & n = odd \\ 0, & n = \text{even} \end{cases}$$

$$W(f) = \sum_{n=-\infty}^{n=\infty} c_n \delta(f - nf_0)$$

- Line spectrum of periodic functions




**Magnitudes of FS coefficients, $C_n$, of the periodic signal w(t)**



One Theorem:

## Line Spectra for A Periodic Signal

- If *w(t)* is a periodic signal with period $T_0$ , then its spectrum (Fourier transform) is:

$$W(f) = \sum_{n=-\infty}^{n=\infty} X_n \delta(f - nf_0)$$

- where $f_0 = 1/T_0$ , and $X_n$ are complex exponential FS coefficients, and $\delta$ is a unit impulse function.

# Another Theorem:
# **FS Coefficients of A Periodic Signal**

- If *w(t)* is a periodic signal with period , $T_0$ , and

$$w(t) = \sum_{n=-\infty}^{n=\infty} v(t - nT_0) = \sum_{n=-\infty}^{n=\infty} X_n e^{jn\omega_0 t}$$

where

$$v(t) = \begin{cases} w(t), & |t| < \frac{T_0}{2} \\ 0, & t \quad elsewhere \end{cases}$$

then $X_n = f_0 V(nf_0)$, $V(f) = FT\{v(t)\}$, and $f_0 = 1/T_0$ .



- Please read yourself
  - Text, page 50, Example 2.8, Figure 2.22
  - Text, page 52, Example 2.9, Figure 2.24

# Fourier Transform

- Definition

$$\begin{cases} x(t) = \int_{-\infty}^{\infty} X(f) e^{j2\pi ft} df & IFT \\ X(f) = \int_{-\infty}^{\infty} x(t) e^{-j2\pi ft} dt & FT \end{cases}$$

- Comment:
  - Compare the pair of formulae with that of FS (the next slide)
  - Note the similarity



# Recall:
# Complex Exponential Fourier Series…

$$x(t) = \sum_{n=-\infty}^{\infty} X_n e^{jn\omega_0 t} \qquad t_0 \le t \le t_0 + T_0$$

$$X_n = \frac{1}{T_0} \int_{t_0}^{t_0+T} x(t) e^{-jn\omega_0 t} dt$$

# Fourier Transform…

- Dirichlet's Conditions

  1. Single-valued with a finite number of maxima and minima
  2. A finite number of discontinuities in any finite time interval
  3. Absolutely integrable, i.e., $\int_{-\infty}^{\infty} |x(t)| dt < \infty$

  They are sufficient conditions for a signal to have a FT.



# Fourier Transform…

- Loosely speaking, from Figure 2.12 (page 29), one can see
  - When $T_0 \uparrow \Rightarrow \frac{1}{T_0} \downarrow, \Rightarrow$ line spectra, denser
  - When $T_0 \rightarrow \infty,$ line spectra → continuous
  - Implies: FT of unit rectangle function → sinc function

* Beginning of Ch3 in text contains a good description on this.

## Amplitude & Phase Spectra

- $x(t) \overset{FT}{\leftrightarrow} X(f)$ Spectrum

$$X(f) = |X(f)| e^{j\theta(f)}$$

Complex in general, even *x(t)* is real.

- $|X(f)|$ : Amplitude Spectrum
- $\theta(f) = \angle X(f)$ : Phase Spectrum



## Amplitude & Phase Spectra: Symmetry

- If x(t) is real, $|X(f)|$ is even

i.e., $|X(f)| = |X(-f)|$

$\theta(f)$ is odd

i.e., $\theta(f) = -\theta(-f)$

just similar to the symmetry of FS Coefficients.

## Amplitude & Phase Spectra: Symmetry …

- $$\because R_e X(f) = \int_{-\infty}^{\infty} x(t)\cos(2\pi f t)dt$$ Even w.r.t. *f*

$$I_m X(f) = \int_{-\infty}^{\infty} x(t)\sin(2\pi f t)dt$$ *Odd* w.r.t. *f*

$$|X(f)| = \sqrt{[R_e X(f)]^2 + [I_m X(f)]^2}$$ Even w.r.t. *f*

$$\theta(f) = \underbrace{arctg \underbrace{\frac{I_m X(f)}{R_e X(f)}}_{odd}}_{odd}$$ *Odd* w.r.t. *f*



## Symmetry Properties …

- If *x(t)* is <u>real and even</u>, [*x(t)* = *x(-t)*, even of *t*]

then *X(f)* is <u>real & even</u> [*X(f)* = *X(-f)* <u>even of *f*</u>]

$$\because \text{Re } X(f) = \int_{-\infty}^{\infty} \underbrace{x(t)\cos(2\pi f t)}_{even\ of\ t} dt, \qquad even \quad of \quad f$$

$$\text{Im } X(f) = \int_{-\infty}^{\infty} \underbrace{x(t)\sin(2\pi f t)}_{odd\ of\ t} dt = 0$$

# Symmetry Properties …

- If *x(t)* is real and odd, [*x(t)* = -*x(-t)*, odd of *t*]

  then *X(f)* is imaginary, odd of *f*.

$$\because \ \operatorname{Re} X(f) = \int_{-\infty}^{\infty} \underbrace{x(t)\cos(2\pi f t)}_{odd\ of\ t} dt = 0$$

$$\operatorname{Im} X(f) = \int_{-\infty}^{\infty} \underbrace{x(t)\sin(2\pi f t)}_{even\ of\ t} dt, \quad odd \quad of \quad f$$



# Example

- $x(t) = A\prod\left(\frac{t-t_o}{\tau}\right)$

$$\frac{t}{\tau} = \frac{1}{2} \Rightarrow t = \frac{\tau}{2}$$

$$-\frac{t}{\tau} = \frac{1}{2} \Rightarrow t = -\frac{\tau}{2}$$

- $\Rightarrow$ Consider $t_0 = 0$ temporarily

$$FT\{X(T)\} = A\left[\int_{-\frac{\tau}{2}}^{\frac{\tau}{2}} \cos(2\pi ft)dt - j\int_{-\frac{\tau}{2}}^{\frac{\tau}{2}} \sin(2\pi ft)dt\right]$$

$$= 2A\frac{\sin(2\pi ft)|_0^{\frac{\tau}{2}}}{2\pi f} = A\frac{\sin(2\pi f\frac{\tau}{2})}{\pi f}$$

$$= A\tau\frac{\sin(\pi f\tau)}{\pi f\tau} = A\tau \sin c(\pi f\tau)$$



After using the shift property of FT

i.e., (time-delay property of FT (a few slides later))

we can have:

$$X(f) = A\tau \sin c(\pi f\tau)e^{-j2\pi ft_0}$$

# Convolution

- Definition

$$x(t) = x_1(t) * x_2(t)$$
$$= \int_{-\infty}^{\infty} x_1(\lambda) x_2(t-\lambda) d\lambda$$
$$= \int_{-\infty}^{\infty} x_1(t-\lambda) x_2(\lambda) d\lambda$$

A special type of integration.

t: is a parameter as far as the integration is concerned.



# Convolution …

- For a linear system: x(t) → [h(t)] → y(t)

$$y(t) = h(t) * x(t)$$

- The integrand is formed from $x_1$ and $x_2$ by three operations:
  1. Time reversal to $x_1(-\lambda)$ or $x_2(-\lambda)$
  2. Time shifting to obtain $x_1(t-\lambda)$ or $x_2(t-\lambda)$
  3. Multiplication of $x_1(\lambda)$ and $x_2(t-\lambda)$ or $x_1(t-\lambda)$ and $x_2(\lambda)$

# Convolution …

- Graphical analysis to determine
  - Different stages
  - Different integration limits
  - Not an easy job
- Fourier Transform Technique
  - Make it easier in the transformed domain
  - Also called frequency domain



# Convolution Theorem

$$x_1(t) * x_2(t) \leftrightarrow X_1(f)X_2(f)$$

where

$$x_1(t) \leftrightarrow X_1(f)$$

$$x_2(t) \leftrightarrow X_2(f)$$

## Linearity of FT (Superposition Theorem)

- $a_1 x_1(t) + a_2 x_2(t) \leftrightarrow a_1 X_1(f) + a_2 X_2(f)$

  where $a_1, a_2$ are constants

  $$x_1(t) \leftrightarrow X_1(f)$$
  $$x_2(t) \leftrightarrow X_2(f)$$

- Proof: From definition of FT



## Time-Delay Theorem

- $x(t - t_0) \leftrightarrow X(f)e^{-j2\pi f t_0}$

  where $x(t) \leftrightarrow X(f)$

- Proof: From definition of FT

## Scale-Change Theorem

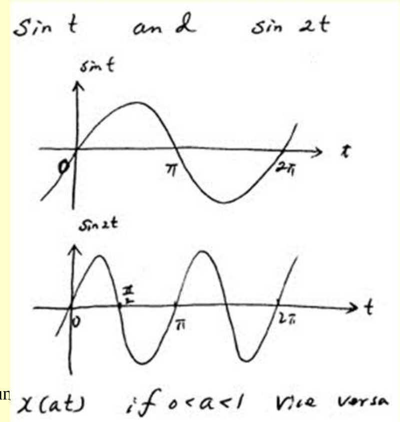
- $$x(at) \leftrightarrow \frac{1}{|a|} X\left(\frac{f}{a}\right)$$
- Proof: if a > 0
  if a < 0

(Using variable substitute and def. of FT)



## Scale-Change Theorem…

- Meaning: $x(at)$ , if *a > 1*, the diagram (graph) of *x(at)* is shrinked by *a* time.
- *Sin t* and *Sin 2t*



## Duality Theorem

- If $x(t) \leftrightarrow X(f)$
  then $X(t) \leftrightarrow x(-f)$
- ***Proof***: From definition

$$FT\{X(t)\} = \int_{-\infty}^{\infty} X(t)e^{-j2\pi ft}dt = \int_{-\infty}^{\infty} X(t)e^{-j2\pi(-f)t}dt$$

$$= x(-f)$$

$$\uparrow$$

$$IFT$$

$$-f = f'$$



## Frequency Translation Theorem

- $x(t)e^{j2\pi f_0 t} \leftrightarrow X(f - f_0)$

- Proof: From definition of IFT

# Modulation Theorem

$$x(t)\cos(2\pi f_0 t) \leftrightarrow \frac{1}{2}X(f - f_0) + \frac{1}{2}X(f + f_0)$$

- Proof:

$$x(t)\cos(2\pi f_0 t) = x(t)\cdot\frac{1}{2}\cdot\left[e^{-j2\pi f_0 t} + e^{j2\pi f_0 t}\right]$$

Using Frequency translation theorem



# Differentiation Theorem

$$\frac{d^n x(t)}{dt^n} \leftrightarrow (j2\pi f)^n X(f)$$

# Integration Theorem

$$\int_{-\infty}^{t} x(\lambda)d\lambda \leftrightarrow \frac{1}{j2\pi f}X(f) + \frac{1}{2}X(0)\delta(f)$$



# Singularity Functions

- An important subclass of aperiodic signals.
- Two will be introduced:
  - $\delta(t)$ : the unit impulse function.
  - $u(t)$ : the unit step function.

# The Unit Impulse Function "the delta function"

- Definition 1: $\int_{-\infty}^{\infty} x(t)\delta(t-t_0)dt = x(t_0) = x(t)\big|_{t=t_0}$

  *x(t)* is a continuous function at t = $t_0$
  This property is called the ***sifting property***

- Definition 2:

$$\begin{cases} \int_{t_1}^{t_2} \delta(t-t_0)dt = 1 & t_1 < t_0 < t_2 \\ \delta(t-t_0) = 0 & t \neq t_0 \end{cases}$$



# The Unit Impulse Function "the delta function"…

- Definition 3:

$$\lim_{\varepsilon \to 0} f(t) = \delta(t)$$


**Figure 2.13** Unit impulse and its approximation.

- Any signal function having a unit area and zero width in the limit as some parameter approaches zero is a suitable representation for $\delta(t)$ .
- Examples: triangle function.

Time domain Freq. domain

$$\begin{cases} \delta(t) \overset{FT}{\leftrightarrow} 1 \\ 1 \overset{FT}{\leftrightarrow} \delta(-f) = \delta(f) \end{cases}$$



Another way to say the above two relations:

$$\begin{cases} \delta(t) = \int_{-\infty}^{\infty} e^{j2\pi ft}\, df \\ \delta(f) = \int_{-\infty}^{\infty} e^{-j2\pi ft}\, dt \end{cases}$$

One more important formula:

$$\phi(t)\delta(t) = \phi(0)\delta(t)$$

# Spectrum of a Sinusoid

- Consider $x(t) = A\cos(2\pi f_0 t)$

FT: $X(f) = \int_{-\infty}^{\infty} \frac{A}{2}\left(e^{j2\pi f_o t} + e^{-j2\pi f_o t}\right)\cdot e^{-j2\pi ft} dt$

$$= \frac{A}{2}\int_{-\infty}^{\infty}\left(e^{-j2\pi(f-f_o)t} + e^{-j2\pi(f+f_o)t}\right)\cdot dt$$



$$\because \quad \delta(f) = \int_{-\infty}^{\infty} e^{-j2\pi ft} dt$$

$$\therefore \quad X(f) = \frac{A}{2}\left[\delta(f - f_0) + \delta(f + f_0)\right]$$

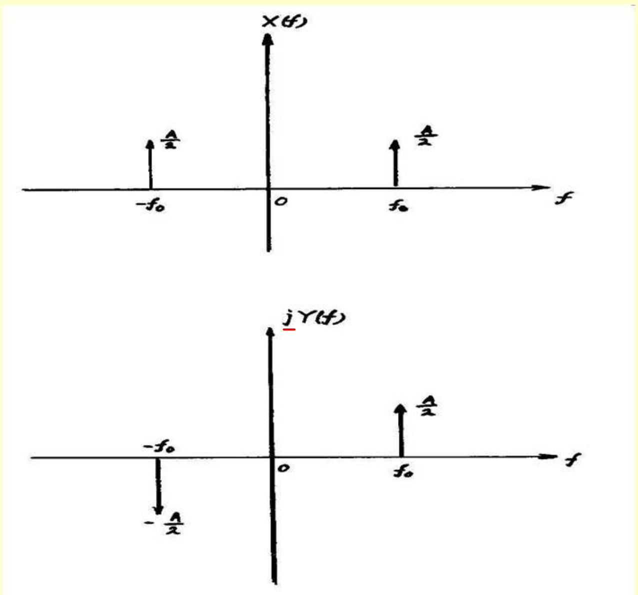
- For $y(t) = A\sin(2\pi f_0 t)$

$$Y(f) = \frac{A}{2j}\left[\delta(f - f_0) - \delta(f + f_0)\right]$$

$$\because Y(f) = \int_{-\infty}^{\infty} \frac{A}{2j}\left[e^{j2\pi f_0 t} - e^{-j2\pi f_0 t}\right] \cdot e^{-j2\pi f t} dt$$



# FT of *cos x* and *sin y*



- Convolution of a function with a unit impulse function = the original function

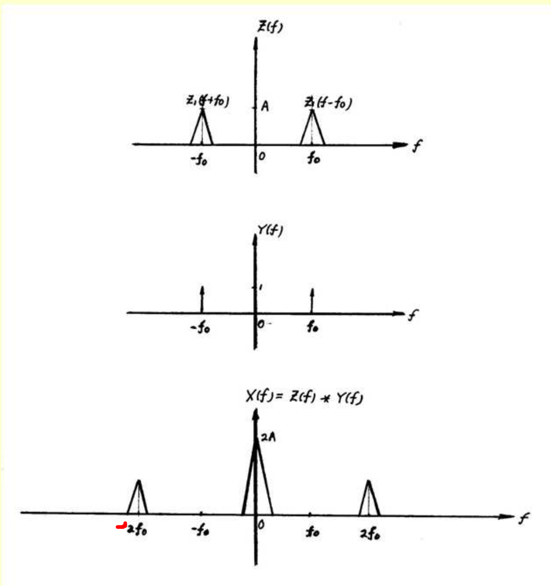
$$\because \qquad x(t) \leftrightarrow X(f)$$
$$and \qquad \delta(t) \leftrightarrow 1$$
$$\therefore \qquad x(t) * \delta(t) \overset{FT}{\leftrightarrow} X(f)$$
$$i.e., \qquad x(t) * \delta(t) = x(t)$$

Similarly, in frequency domain:

$$\Rightarrow \begin{matrix} X(f) * \delta(f) = X(f) \\ X(f) * \delta(f - f_0) = X(f - f_0) \end{matrix}$$



Example



$$\because [Z_1(f-f_0)+Z_1(f+f_0)]*[\delta(f-f_0)+\delta(f+f_0)]$$
$$=Z_1(f-f_0)*\delta(f-f_0)+Z_1(f-f_0)*\delta(f+f_0)$$
$$+Z_1(f+f_0)*\delta(f-f_0)+Z_1(f+f_0)*\delta(f+f_0)$$
$$=2Z_1(f)+Z_1(f-2f_0)+Z_1(f+2f_0)$$



# Unit Step Function *u(t)*

$$u(t)\overset{\Delta}{=}\int_{-\infty}^{t}\delta(\lambda)d\lambda=\begin{cases}0, & t<0\\ 1, & t>0\\ undefined & t=0\end{cases}$$

$$\delta(t)=\frac{du(t)}{dt}$$

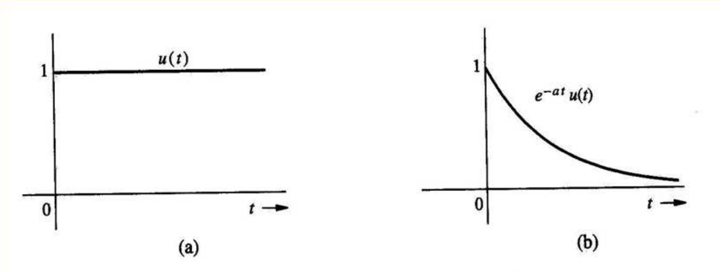



**Figure 2.14** (a) Unit step function $u(t)$. (b) Causal exponential $e^{-at}u(t)$.

- The unit rectangular pulse can be defined as:

$$\prod(t) = u\left(t + \frac{1}{2}\right) - u\left(t - \frac{1}{2}\right)$$



# Multiplication Theorem

$$x_1(t)x_2(t) \leftrightarrow X_1(f) * X_2(f)$$

# Examples

- Example 1

Since, $A\prod\left(\frac{t}{\tau}\right) \leftrightarrow A\tau \sin c(f\tau)$ duality

$$\therefore A\tau \sin c(\tau t) \leftrightarrow A\prod\left(\frac{-f}{\tau}\right)$$

$$= A\prod\left(\frac{f}{\tau}\right)$$



# Examples

- Example 2:

1. $A\delta(t) \leftrightarrow A$ $[or, \delta(t) \leftrightarrow 1]$ linearity

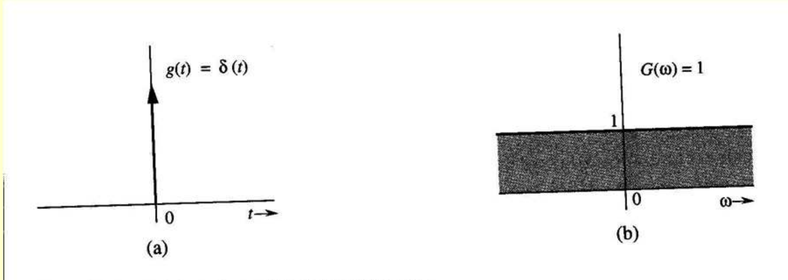



**Figure 3.11** Unit impulse and its Fourier spectrum.

2. $A\delta(t-t_0) \leftrightarrow Ae^{-j2\pi f t_0}$ time delay

3. $A \leftrightarrow A\delta(f)$ $[or, 1 \leftrightarrow \delta(f)]$ linearity

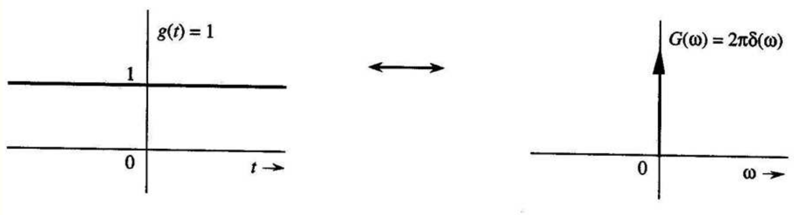



**Figure 3.12** Constant (dc) signal and its Fourier spectrum.



4. $Ae^{j2\pi f_0 t} \leftrightarrow A\delta(f-f_0)$ frequency translation

- Proof:

1. $FT\{\delta(t)\} = \int_{-\infty}^{\infty} \delta(t)e^{-j2\pi f t}dt = e^{-j2\pi f \cdot 0} = 1$

2. $1 \leftrightarrow \delta(-f) = \delta(f),$ $(\delta(f):$ *even function*$)$

## Summary

- Fourier series:
  - Complex exponential series
  - Trigonometric series: compact and quadrature
  - Properties
- Fourier transform
  - Definition
  - Properties
- Special functions: sinc, rect, delta, step
- Others:
  - Orthoganality
  - Dirichlet conditions
  - Some techniques in mathematical manipulation
- Tables 3.1 and Table 3.2