Localization Performance of Coherent MIMO Radar Systems Subject to Phase Synchronization Errors

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***Abstract*—Coherent Multiple-Input Multiple-Output (MIMO) radar with ideal phase synchronization among distributed radar stations has been shown to support high resolution and high accuracy target localization. In this paper, quantitative tools are developed for assessing the effect of phase errors on the localization performance. The hybrid Cramer-Rao bound (HCRB) is derived for the joint estimation of the unknown, deterministic target location and the unknown, random phase errors. It is shown that for the coherent MIMO radar problem, the HCRB is asymptotically tight in the sense that it is equal to the marginal Cramer-Rao bound (CRB). In the marginal CRB, the phase errors are treated as nuisance parameters and are marginalized. The HCRB is shown to follow the CRB obtained in the absence of phase errors, up to a threshold signal-to-noise ratio (SNR), determined by the phase error variance, and the number of mismatched transmitting and receiving sensors. Beyond this point, the HCRB is limited by the variance of the phase errors, and stops decreasing with the SNR. A simple expression for the threshold SNR is derived for symmetrical radar deployments.**

***Index Terms*—MIMO radars, Hybrid CRB, localization.**

I. INTRODUCTION

Multiple-Input Multiple-Output (MIMO) radar systems with widely spread antennas take advantage of the geographical spread of the deployed sensors, offering significant improvement in target parameter estimation capabilities [1]. Coherent MIMO radar systems have been shown to provide high resolution and high accuracy target localization at high signal to noise ratio (SNR) [2]. A Cramer-Rao bound (CRB) [3] based performance evaluation is provided in [2] for coherent and non-coherent MIMO radar systems. In both cases, the localization accuracy improves with the order of the product of the numbers of transmit and receive radars. Nonetheless, coherent processing provides higher estimation accuracies when compared with the non-coherent case, proportional to the ratio between the signal carrier frequency and the signal effective bandwidth. The latter performance gain comes with the challenge of attaining phase synchronization in a distributed system. Errors introduced into the system parameters by phase synchronization inaccuracies will result in an increase of the localization mean-square error (MSE). Quantifying the effects of phase errors on the localization in coherent MIMO radar is the topic of this paper.

In the localization problem, the unknown target location is a deterministic quantity, while the phase errors are modeled as unknown random parameters. Since we are not interested in the estimation of the phase errors, they serve as nuisance parameters. Assuming that the a priori probability density function (PDF) of the phase errors is known, one way to handle the nuisance parameters is to marginalize them. Marginalization means the computation of the marginal PDF of the observations by averaging out the unknown random parameters. The CRB obtained after marginalizing the unknown random parameters is often referred to as marginal CRB. It is known that the marginal CRB is asymptotically tight, i.e., it is achieved by the maximum likelihood estimator (MLE) for large SNR [6]. However, the marginal PDF of the observations is often difficult or impossible to derive. This difficulty can be circumvented by the hybrid CRB (HCRB), introduced by Rockah and Schultheiss [4]. This bound has been applied to the problem of passive source localization with phase synchronization errors [5]. For a wide range of problems, the HCRB is mathematically tractable, offering closed-form solutions to support performance analysis. In HCRB, the unknown parameters include both the unknown deterministic parameters as well as the unknown random parameters. On the downside, it is known that HCRB is not, in general, a tight bound. Recently, a necessary and sufficient condition was formulated under which the HCRB of nonrandom parameters is equal to the marginal CRB of the same parameters [7].

Coherent MIMO radar with phase errors has been the topic of some recent publications. In [8], we have derived an analytical expression of the HCRB for target localization in coherent MIMO radar systems with phase synchronization errors. We have shown that the HCRB can be expressed as a sum of two terms: a term equal to the CRB without phase errors and a term due to the phase errors. Localization performance analysis of coherent MIMO radar systems by marginalization of phase errors has been studied in [9]. Since computation of the marginal PDF cannot be carried out analytically, the work focuses on numerical analysis of the marginal CRB. In the current paper, we extend [8] by showing that the HCRB for the MIMO radar problem is tight, and develop an insightful closed-form expression for a special case.

The paper is organized as follows: The system model is introduced in Section II; the HCRB is introduced and its properties discussed in Section III; performance analysis with phase errors is presented in Section IV, by deriving the threshold point and asymptotic limit for the case of symmetrical

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radar placement. Finally, Section V concludes the paper.

II. SYSTEM MODEL

A point target is assumed to have complex reflectivity $\vartheta = \vartheta_{\mathrm{Re}} + j\vartheta_{\mathrm{Im}}$ and be located in a two dimensional plane at coordinates $X_0 = (x_0, y_0)$. Consider a set of M transmitting stations and N receiving stations, widely distributed over a given geographical area, and time and phase synchronized. A set of orthogonal waveforms is transmitted, with the lowpass equivalents $s_k(t)$, $k = 1, \ldots, M$, and effective bandwidths β [10]. The signals are narrowband in the sense that for a carrier frequency of f_c, the narrowband signal assumption implies $\beta^2 / f_c^2 \ll 1$.

In the derivation of the CRB in [2], perfect phase synchronization was assumed. In practice, synchronization errors exist, modeled here as zero mean Gaussian random variables with common variance σ_Δ^2 and denoted by $\boldsymbol{\phi} = \left[\phi_{t_1}, \phi_{t_2}, ..., \phi_{t_M}, \phi_{r_1}, \phi_{r_2}, ..., \phi_{r_N}\right]^T$, where ϕ_{t_k} and ϕ_{r_ℓ} are phase errors at transmitting station k and receiving station ℓ, respectively. The phase errors introduced by the different stations are assumed to be statistically independent with PDF of the form

$$p_{\boldsymbol{\phi}}(\boldsymbol{\phi}) \propto \exp\left\{-\frac{1}{2\sigma_\Delta^2}\left(\sum_{k=1}^{M}\phi_{t_k} + \sum_{\ell=1}^{N}\phi_{r_\ell}\right)\right\}. \tag{1}$$

The vector of unknown parameters is defined by $\boldsymbol{\theta} = \left[\boldsymbol{\theta}_d^T, \boldsymbol{\phi}^T\right]^T$, where $\boldsymbol{\theta}_d = [x_0, y_0, \vartheta_{\mathrm{Re}}, \vartheta_{\mathrm{Im}}]^T$ denotes the deterministic unknowns and $\boldsymbol{\phi}$ denotes the random unknowns. The estimation process is based on the signals observed at the receiving sensors. The signal received at sensor ℓ is a superposition of the transmitted signals, reflected from the target, and is given by

$$r_\ell(t) = \sum_{k=1}^{M}\vartheta s_k(t - \tau_{\ell k})\,\eta_{\ell k} + n_\ell(t), \tag{2}$$

where $\eta_{\ell k}$ accounts for the phase information and has the value of $\eta_{\ell k} = \exp(-j2\pi f_c \tau_{\ell k})\exp\left(-j\left(\phi_{t_k} + \phi_{r_\ell}\right)\right)$. The noise $n_\ell(t)$ is assumed to be circularly symmetric, zero-mean, complex Gaussian, spatially and temporally white with autocorrelation function $\sigma_n^2\delta(\tau)$. The propagation time, $\tau_{\ell k}$, is the sum of the time delays from station k to the target and from the target to station ℓ, and may be expressed as

$$\begin{aligned}\tau_{\ell k} = \frac{1}{c}\Bigg(&\sqrt{(x_{tk} - x_o)^2 + (y_{tk} - y_o)^2} \\ &+\sqrt{(x_{r\ell} - x_o)^2 + (y_{r\ell} - y_o)^2}\Bigg),\end{aligned} \tag{3}$$

where c denotes the speed of light, (x_{tk}, y_{tk}) denotes the location of transmitting radar k and $(x_{r\ell}, y_{r\ell})$ denotes the location of receiving radar ℓ. The following vector notation is introduced: $\boldsymbol{\tau} = [\tau_{11}, \tau_{12}, ..., \tau_{\ell k}, ..., \tau_{NM}]^T$, $\mathbf{r} = [r_1(t), \ldots, r_N(t)]^T$, $Q = MN$, $L = M + N$.

The received signals are separated at the receiver by exploiting the orthogonality between the transmitted waveforms. This orthogonality is assumed to be maintained between the received signals. The vector of unknown parameters for the observations $r_\ell(t)$ in (2) is expressed as a function of the time delays $\boldsymbol{\tau}$ rather than a function of the unknown location (x_0, y_0) (as seen in (3)); i.e., the vector of unknown parameters is denoted by $\boldsymbol{\kappa} = \left[\boldsymbol{\kappa}_d^T, \boldsymbol{\phi}^T\right]^T$, with $\boldsymbol{\kappa}_d = \left[\boldsymbol{\tau}^T, \vartheta_{\mathrm{Re}}, \vartheta_{\mathrm{Im}}\right]^T$.

III. HCRB

In this section, an analytical expression is derived for the HCRB, and its relationship with the marginal CRB is discussed.

A. HCRB with Phase Synchronization Errors

The hybrid CRB provides a lower bound for the MSE of any unbiased estimator for an unknown parameter(s), where the parameters are partially deterministic and partially random [4]. Given an observation vector $\mathbf{r}$ and an unknown vector $\boldsymbol{\theta}$, the unbiased estimate $\hat{\boldsymbol{\theta}}$ satisfies the following inequality [4]:

$$E\left\{\left(\hat{\boldsymbol{\theta}} - \boldsymbol{\theta}\right)\left(\hat{\boldsymbol{\theta}} - \boldsymbol{\theta}\right)^T\right\} \geq \left(\mathbf{J}_H(\boldsymbol{\theta})\right)^{-1}. \tag{4}$$

The matrix $\mathbf{J}_H$ is the hybrid Fisher Information matrix (HFIM) given by

$$\mathbf{J}_H(\boldsymbol{\theta}) = E_{\mathbf{r},\boldsymbol{\phi}}\left\{\frac{\partial^2 \ln p_{\mathbf{r},\boldsymbol{\phi}}(\mathbf{r}, \boldsymbol{\phi}\,|\boldsymbol{\theta}_d)}{\partial\boldsymbol{\theta}\partial\boldsymbol{\theta}^T}\right\}, \tag{5}$$

where $p_{\mathbf{r},\boldsymbol{\phi}}(\mathbf{r}, \boldsymbol{\phi}\,|\boldsymbol{\theta}_d)$ is the conditional, joint PDF of the observations and the random vector $\boldsymbol{\phi}$. In order to derive the HFIM in (5), the conditional joint PDF $p(\mathbf{r}|\boldsymbol{\theta}_d, \boldsymbol{\phi})$ is required. Instead, for the signal model in (2), we can write the conditional joint PDF of the observations parametrized by $\boldsymbol{\kappa}$:

$$\begin{aligned}&p_{\mathbf{r}}(\mathbf{r}|\boldsymbol{\kappa}_d, \boldsymbol{\phi}) \propto \exp \\ &\left\{-\frac{1}{\sigma_n^2}\sum_{\ell=1}^{N}\int_T\left|r_\ell(t) - \sum_{k=1}^{M}\vartheta s_k(t - \tau_{\ell k})\,\eta_{\ell k}\right|^2 dt\right\}.\end{aligned} \tag{6}$$

The relationship between the elements of the unknown vectors $\boldsymbol{\theta}$ and $\boldsymbol{\kappa}$ are given by $\kappa_j = g_j(\boldsymbol{\theta})$ in (3). The HFIM $\mathbf{J}_H(\boldsymbol{\kappa})$ can be derived from (6). The *chain rule* [3] can be used to find $\mathbf{J}_H(\boldsymbol{\theta})$ is given as

$$\mathbf{J}_H(\boldsymbol{\theta}) = \mathbf{P}\left(\mathbf{J}_H(\boldsymbol{\kappa})\right)\mathbf{P}^T, \tag{7}$$

where the elements of the matrix $\mathbf{P}$ are $[\mathbf{P}]_{i,j} = \partial g_j(\boldsymbol{\theta})/\partial\theta_i$. Using the conditional PDF $p(\mathbf{r}|\boldsymbol{\kappa}_d, \boldsymbol{\phi})$ in (6) and the Gaussian distribution of the phase errors in (1), the HFIM $\mathbf{J}_H(\boldsymbol{\theta})$, defined in (5), is derived in [8]. It can be shown that the expression of the HCRB, $\mathbf{C}_H(x_0, y_0)$, can be written [8],

$$\begin{aligned}\mathbf{C}_H(x_0, y_0) &= \mathbf{J}_0^{-1} + \left[\mathbf{J}_0\boldsymbol{\Lambda}_\Delta^{-1}\mathbf{J}_0 - \mathbf{J}_0\right]^{-1} \\ &= \mathbf{C}_{CRB}(x_0, y_0)\big|_{\boldsymbol{\phi}=\mathbf{0}} + \boldsymbol{\Delta}_{CRB},\end{aligned} \tag{8}$$

where $\mathbf{C}_{CRB}(x_0, y_0)\big|_{\boldsymbol{\phi}=\mathbf{0}} = \mathbf{J}_0^{-1}$ is the CRB in the absence of phase errors, the matrix $\Delta_{CRB} = \left[\mathbf{J}_0^{-1}\boldsymbol{\Lambda}_\Delta^{-1}\mathbf{J}_0 - \mathbf{J}_0\right]^{-1}$ represents the increment in the bound due to phase synchronization errors, and the matrix $\mathbf{J}_0$ can be expressed as [8]

$$\mathbf{J}_0 = \mu_0\mathbf{D}^T\left(\mathbf{I} - \frac{1}{MN}\left(\mathbf{1}\mathbf{1}^T\right)_{Q\times Q}\right)\mathbf{D}, \tag{9}$$

where $\mathbf{1} = [1, 1, , ..., 1]^T$. For the MIMO radar problem, the matrix $\mathbf{D}$ is defined in Appendix A (see (20)). The matrix $\mathbf{\Lambda}_\triangle$ is defined

$$\mathbf{\Lambda}_\triangle = \mu_0 \sum_{m=1}^{3} \mu_m \mathbf{B}_m, \tag{10}$$

with matrices $\mathbf{B}_m$ given in Appendix A (see (23), (24), and (25)). The constants μ_m, $m = 0, ..., 3$, depend on the phase synchronization error variance $\sigma_\triangle^2$ and the system parameters, as follows:

$$\begin{aligned} \mu_0 &= \tfrac{8\pi^2 f_c^2 \operatorname{snr}}{c^2}, \\ \mu_1 &= -\tfrac{2 \operatorname{snr} \sigma_\triangle^2}{M\left(1+2 \operatorname{snr} \sigma_\triangle^2 N\right)} - \tfrac{2 \operatorname{snr} \sigma_\triangle^2}{N\left(1+2 \operatorname{snr} \sigma_\triangle^2 M\right)}, \\ \mu_2 &= \tfrac{2 \operatorname{snr} \sigma_\triangle^2 N^2}{M\left(1+2 \operatorname{snr} \sigma_\triangle^2 N\right)}, \\ \mu_3 &= \tfrac{2 \operatorname{snr} \sigma_\triangle^2 M^2}{N\left(1+2 \operatorname{snr} \sigma_\triangle^2 M\right)}, \end{aligned} \tag{11}$$

and

$$\mu_3 = \tfrac{2 \operatorname{snr} \sigma_\triangle^2 M^2}{N\left(1+2 \operatorname{snr} \sigma_\triangle^2 M\right)},$$

with $\operatorname{snr} = |\vartheta|^2 / \sigma_n^2$ representing the SNR.

B. Marginal CRB and the HCRB

The CRB on the deterministic unknown parameters $\boldsymbol{\theta}_d$ is given by [3]

$$E\left\{ \left(\hat{\boldsymbol{\theta}}_d - \boldsymbol{\theta}_d\right) \left(\hat{\boldsymbol{\theta}}_d - \boldsymbol{\theta}_d\right)^T \right\} \geq E_{\mathbf{r}} \left\{ \frac{\partial^2 \ln p_{\mathbf{r}} \left(\mathbf{r} \,|\boldsymbol{\theta}_d \right)}{\partial \boldsymbol{\theta} \partial \boldsymbol{\theta}^T} \right\}. \tag{12}$$

When $\mathbf{r}$ depends on unknown random parameters $\boldsymbol{\phi}$, those can be eliminated by marginalization of the PDF, i.e., $p_{\mathbf{r}} \left(\mathbf{r} \,|\boldsymbol{\theta}_{nr} \right) = \int_{\mathbb{R}^L} p_{\mathbf{r},\boldsymbol{\phi}} \left(\mathbf{r}, \boldsymbol{\phi} \,|\boldsymbol{\theta}_d \right) p_{\boldsymbol{\phi}} \left(\boldsymbol{\phi}\right) d\boldsymbol{\phi}$. In [7], a necessary and sufficient condition for the HCRB (8) to be equal to the marginal CRB (12) is shown to be

$$\frac{\partial^2 \ln p_{\mathbf{r},\boldsymbol{\phi}} \left(\mathbf{r}, \boldsymbol{\phi} \,|\boldsymbol{\theta}_d \right)}{\partial \boldsymbol{\phi} \partial \boldsymbol{\theta}_d^T} = \mathbf{F}_{\boldsymbol{\theta}_d,\boldsymbol{\phi}} \mathbf{F}_{\boldsymbol{\phi}}^{-1} \frac{\partial^2 \ln p_{\mathbf{r},\boldsymbol{\phi}} \left(\mathbf{r}, \boldsymbol{\phi} \,|\boldsymbol{\theta}_d \right)}{\partial \boldsymbol{\phi} \partial \boldsymbol{\phi}^T}, \tag{13}$$

where the matrix $\mathbf{F}_{\boldsymbol{\theta}_d,\boldsymbol{\phi}}$ is

$$\mathbf{F}_{\boldsymbol{\theta}_d,\boldsymbol{\phi}} = E_{\mathbf{r},\boldsymbol{\phi}} \left\{ \frac{\partial^2 \ln p_{\mathbf{r},\boldsymbol{\phi}} \left(\mathbf{r}, \boldsymbol{\phi} \,|\boldsymbol{\theta}_d \right)}{\partial \boldsymbol{\phi} \partial \boldsymbol{\theta}_d^T} \right\}, \tag{14}$$

and the matrix $\mathbf{F}_{\boldsymbol{\phi}}$ is

$$\mathbf{F}_{\boldsymbol{\phi}} = E_{\mathbf{r},\boldsymbol{\phi}} \left\{ \frac{\partial^2 \ln p_{\mathbf{r},\boldsymbol{\phi}} \left(\mathbf{r}, \boldsymbol{\phi} \,|\boldsymbol{\theta}_d \right)}{\partial \boldsymbol{\phi} \partial \boldsymbol{\phi}^T} \right\}. \tag{15}$$

Given (1) and (6), the logarithm of the conditional joint PDF is

$$\begin{aligned} \ln p_{\mathbf{r},\boldsymbol{\phi}} \left(\mathbf{r}, \boldsymbol{\phi} \,|\boldsymbol{\kappa}_d \right) &= c_1 - \frac{1}{2\sigma_\triangle^2} \left(\sum_{k=1}^{M} \phi_{t_k} + \sum_{\ell=1}^{N} \phi_{r_\ell} \right) \\ &\quad - \frac{1}{\sigma_n^2} \sum_{\ell=1}^{N} \int_T \left| r_\ell(t) - \sum_{k=1}^{M} \vartheta s_k \left(t - \tau_{\ell k}\right) \eta_{\ell k} \right|^2 dt, \end{aligned}$$

where c_1 is a constant. Specific expression of the matrices $\mathbf{F}_{\boldsymbol{\theta}_d,\boldsymbol{\phi}}$ and $\mathbf{F}_{\boldsymbol{\phi}}$ for our case are derived in Appendix B (see (26) and (29)). Both are independent of the random nuisance parameters vector $\boldsymbol{\phi}$. Therefore, $\mathbf{F}_{\boldsymbol{\phi}}^{-1} = \left[\frac{\partial^2 \ln p_{\mathbf{r},\boldsymbol{\phi}} (\mathbf{r}, \boldsymbol{\phi} |\boldsymbol{\theta}_d)}{\partial \boldsymbol{\phi} \partial \boldsymbol{\phi}^T} \right]^{-1}$, $\mathbf{F}_{\boldsymbol{\theta}_d,\boldsymbol{\phi}} = \frac{\partial^2 \ln p_{\mathbf{r},\boldsymbol{\phi}} (\mathbf{r}, \boldsymbol{\phi} |\boldsymbol{\theta}_d)}{\partial \boldsymbol{\phi} \partial \boldsymbol{\theta}_d^T}$, and the equality condition in (13) holds. It follows that the HCRB provides a tight bound, achieved by the MLE asymptotically at high SNR.

IV. PERFORMANCE ANALYSIS

The expression for the HCRB is given in (8). From this expression, the increase in the bound due to the phase errors is $\mathbf{\Delta}_{CRB}$. The matrix $\mathbf{\Delta}_{CRB}$ is a function of the locations of the radar stations through the matrices $\mathbf{B}_m$ in (10). The μ_m coefficients incorporate the effect of other system parameters, such as SNR, phase error variance $\sigma_\triangle^2$, and the number of mismatched transmitting and receiving radar stations. The manner in which the system parameters and the phase synchronization error variance affect the performance is not readily discerned from (9) and (10). Fig. 1 shows numerical examples of the HCRB and the MLE for several phase error variances. The figure was generated for $M = 4$, $N = 4$ and $\sigma_\triangle^2 = [0, 0.0025, 0.04, 0.09]\, \pi^2$. It is observed that for a given phase error variance, as the SNR increases beyond a threshold value, the HCRB, as well as the MLE, become limited by the phase errors. To specify a value, let the threshold be the SNR value for which the component of the HCRB not related to phase errors is equal to the component related to phase errors, i.e., the threshold SNR is found from $\mathbf{C}_{CRB} \left(x_0, y_0\right)|_{\boldsymbol{\phi}=\mathbf{0}} = \mathbf{\Delta}_{CRB}$. From the figure it is observed that the threshold occurs at lower SNR for higher phase error variance. The tightness of the HCRB is clearly displayed in the figure.

A simple closed form expression of the HCRB (8) is obtained for the optimal radar placement sets discussed in [2]. For these cases, it can be shown that $\mathbf{J}_0 = \mu_0 MN\mathbf{I}$, $\mathbf{B}_1 = \mathbf{0}$, $\mathbf{B}_2 = \mu_o \mu_2 \frac{M^2}{2} \mathbf{I}$, and $\mathbf{B}_3 = \mu_o \mu_3 \frac{N^2}{2} \mathbf{I}$. Applying these values to (8) results in

$$\mathbf{C}_H \left(x_0, y_0\right) = \frac{c^2}{8\pi^2 f_c^2} \frac{\left[1 + 2(M+N) \operatorname{snr} \sigma_\triangle^2 + 2MN \operatorname{snr}^2 \sigma_\triangle^4\right]}{\operatorname{snr} MN \left[1 + (M+N) \operatorname{snr} \sigma_\triangle^2\right]} \mathbf{I}. \tag{16}$$

The contribution of the phase error can be expressed as

$$\Delta_{CRB} = \frac{1}{\mu_o MN} \frac{\operatorname{snr} \sigma_\triangle^2 \left[2MN \operatorname{snr} \sigma_\triangle^2 + (M+N)\right]}{\left[1 + (M+N) \operatorname{snr} \sigma_\triangle^2\right]} \mathbf{I}. \tag{17}$$

The asymptotic lower limit is defined as the value of $\mathbf{C}_H \left(x_0, y_0\right)$ for $\operatorname{snr} \to \infty$:

$$\lim_{\operatorname{snr} \to \infty} \mathbf{C}_H \left(x_0, y_0\right) = \frac{c^2}{8\pi^2 f_c^2} \frac{\sigma_\triangle^2}{(M+N)} \mathbf{I}. \tag{18}$$

This expression shows the limiting of the bound by the phase errors, independent of the SNR. The expression (17) can also be used to compute the threshold at which the effect of the phase errors becomes dominant, specifically, $\Delta_{CRB} = \mathbf{C}_{CRB} \left(x_0, y_0\right)|_{\mathbf{\Delta}\boldsymbol{\phi}=\mathbf{0}} = \mathbf{J}_o^{-1} = \frac{1}{\mu_o MN} \mathbf{I}$. It is straightforward to show that the threshold point is defined by

$$\operatorname{snr}_{th} \geq \frac{1}{2\sigma_\triangle^2 \sqrt{MN}}. \tag{19}$$

This expression clearly indicates that the higher the phase errors, the lower the SNR at which the performance becomes limited by them. The values of the lower limit and threshold

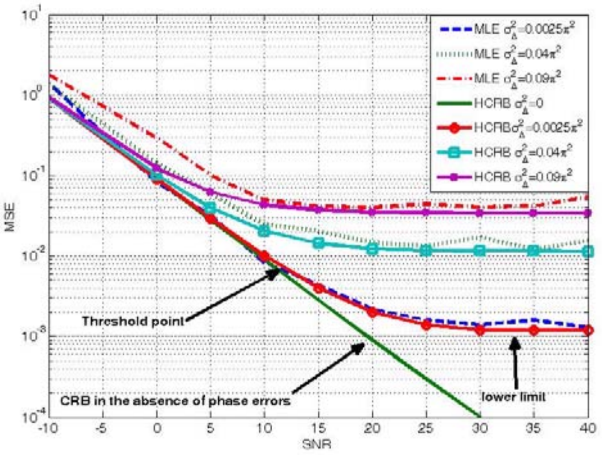

Fig. 1. The HCRB and performance of the MLE for a coherent MIMO radar system with $M = N = 4$ and $\sigma_\Delta^2 = (0, 0.0025\pi^2, 0.04\pi^2, 0.09\pi^2)$.

point can serve as design tools in the system phase calibration, for a given phase synchronization error variance σ_Δ^2.

V. CONCLUSIONS

Localization performance analysis has been studied for distributed coherent MIMO radar where the radar stations are subject to phase errors. A closed-form expression for the HCRB of target localization has been derived for symmetrical sensor layouts, capturing the impact of synchronization errors on the achievable target localization accuracy. In particular, a threshold point has been identified and determined analytically. This point defines the phase error variance for which the synchronization error penalty term is equal to or larger than the CRB in the absence of phase errors. It has been shown to be a function of the product of the number of radars and the SNR. As the SNR increases, the HCRB is asymptotically limited to a value determined by the phase error variance, the number of radars, and the signal carrier frequency. Necessary and sufficient conditions for equality between the HCRB and the marginal CRB have been shown to be met and therefore the HCRB is asymptotically tight at high SNR. The CRB-based lower bounds are appropriate for high SNR and *small errors* and, as such, ignore effects that could lead to *large errors*.

APPENDIX A

The chain rule matrix $\mathbf{D}$ is derived based on the relationship defined in (3) and the chain rule given in (7), resulting in

$$\mathbf{D} = -\frac{1}{c}\begin{bmatrix} \cos\alpha_1 + \cos\gamma_1 & \cdots & \cos\alpha_M + \cos\gamma_N \\ \sin\alpha_1 + \sin\gamma_1 & \cdots & \sin\alpha_M + \sin\gamma_N \end{bmatrix}^T, \tag{20}$$

where α_k and γ_ℓ are the bearing angle of the k-th transmitting radar and the ℓ-th receiving radar with respect to the x axis. Define two matrices,

$$\mathbf{D}_{tx} = \begin{bmatrix} \cos\alpha_1 & \cdots & \cos\alpha_M \\ \sin\alpha_1 & \cdots & \sin\alpha_M \end{bmatrix}^T, \tag{21}$$

and

$$\mathbf{D}_{rx} = \begin{bmatrix} \cos\gamma_1 & \cdots & \cos\gamma_N \\ \sin\gamma_1 & \cdots & \sin\gamma_N \end{bmatrix}^T. \tag{22}$$

The matrices $\mathbf{B}_m$, $m = 1, 2, 3$, in (10) depend on the geographical layout of the radars with respect to the target, and may now be defined as

$$\mathbf{B}_1 = \mathbf{D}^T\left(\mathbf{1}\mathbf{1}^T\right)\mathbf{D}, \tag{23}$$

$$\mathbf{B}_2 = \mathbf{D}_{tx}^T\left(M\mathbf{I} - \left(\mathbf{1}\mathbf{1}^T\right)\right)\mathbf{D}_{tx}, \tag{24}$$

and

$$\mathbf{B}_2 = \mathbf{D}_{rx}^T\left(N\mathbf{I} - \left(\mathbf{1}\mathbf{1}^T\right)\right)\mathbf{D}_{rx}. \tag{25}$$

APPENDIX B

Herein, elements of the matrices $\mathbf{F}_{\boldsymbol{\theta}_d,\boldsymbol{\phi}}$ and $\mathbf{F}_{\boldsymbol{\phi}}$ are derived. For the matrix $\mathbf{F}_{\boldsymbol{\theta}_d,\boldsymbol{\phi}}$ the following expression is obtained:

$$\mathbf{F}_{\boldsymbol{\theta}_d,\boldsymbol{\phi}} = \begin{bmatrix} \mathbf{F}_{\tau,\boldsymbol{\phi}}^T & \mathbf{F}_{\vartheta,\boldsymbol{\phi}}^T \end{bmatrix}, \tag{26}$$

where the matrix $\mathbf{F}_{\tau,\boldsymbol{\phi}}$ is

$$\mathbf{F}_{\tau,\boldsymbol{\phi}} = 4\pi f_c \,\mathrm{snr} \begin{bmatrix} \mathbf{I}_{M\times M} & \cdots & \mathbf{I}_{M\times M} \\ \hline \Pi(1) & \cdots & \Pi(N) \end{bmatrix}^T, \tag{27}$$

with $\Pi(\ell) = \begin{bmatrix} \mathbf{0}_{N\times(\ell-1)} & \mathbf{1}_{N\times 1} & \mathbf{0}_{N\times(N-\ell-1)} \end{bmatrix}$, and the matrix $\mathbf{F}_{\vartheta,\boldsymbol{\phi}}$ is given by

$$\mathbf{F}_{\vartheta,\boldsymbol{\phi}} = \frac{2\,\mathrm{snr}}{|\vartheta|^2}\begin{bmatrix} \vartheta_{\mathrm{Im}} N\mathbf{1}^T & \vartheta_{\mathrm{Im}} M\mathbf{1}^T \\ -\vartheta_{\mathrm{Re}} N\mathbf{1}^T & -\vartheta_{\mathrm{Re}} M\mathbf{1}^T \end{bmatrix}. \tag{28}$$

For the matrix $\mathbf{F}_{\boldsymbol{\phi}}$ we obtain

$$\mathbf{F}_{\boldsymbol{\phi}} = 2\,\mathrm{snr}\begin{bmatrix} N\mathbf{I} & \mathbf{1}\mathbf{1}^T \\ \mathbf{1}\mathbf{1}^T & M\mathbf{I} \end{bmatrix} + \frac{1}{\sigma_\Delta^2}\mathbf{I}_{L\times L}. \tag{29}$$

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