BER Analysis of MPSK Space-Time Block Codes

Chunjun Gao, Alexander M. Haimovich, Debang Lao

Center for Communications and Signal Processing Research, ECE Department,

New Jersey Institute of Technology, Newark, NJ 07102, USA

E-mail: cxg0333@oak.njit.edu, {haimovic,dxl0122}@njit.edu

This work was supported by NSF Award No. CCR 0085846 and AFOSR Grant F49620-00-1-0107

Abstract

Closed-form expressions of bit error rate (BER) are derived for space-time block codes (STBC) based on Alamouti's scheme and utilizing M-ary phase shift keying (MPSK) modulation. The analysis is carried out for the slow, flat Rayleigh fading channel with coherent detection and with non-coherent differential encoding/decoding. The BER expression for coherent detection is exact, while for differential detection it is an approximation for high signal to noise ratio. Numerical results are provided for analysis and simulations for BPSK and QPSK modulations.

Index Terms

Space-time block code, differential detection, coherent detection, bit error rate

I. Introduction

Space-time coding has been a topic of intensive research in recent years. Convolutional, trellis-based space-time codes were first proposed by Tarokh et. al. [1]. A simple space-time block code (STBC) for two transmit antennas was invented by Alamouti [2]. Differential schemes were proposed for cases when either due to rapid changes in the channel or due to the overhead needed to estimate a large number of parameters in a multi-input multi-output (MIMO) system, the channel state information (CSI) is unavailable [3] [4] [5].

Performance analysis of trellis space-time codes and space-time block codes has been traditionally based on the union or other bounds. The union bound is constructed from pairwise error probabilities. In [1], an upper bound was derived for the pairwise error probability of space-time trellis codes. The bound is used to analyze the diversity and coding gains of such codes. An union bound on the symbol error rate (SER) for STBC was obtained in [6]. For *receive* diversity over the Rayleigh fading channel with coherent detection, a closed-form expression of the BER was derived in [7] based on the probability density function (PDF) of the instantaneous signal to noise ratio (SNR). By approximating a MIMO Rayleigh channel as a single-input single-output (SISO) Gaussian channel, an *approximate* expression for the BER of certain STBCs is obtained in [8].

The work presented in this letter is motivated by the observation that for the special case of STBC based on Alamouti's scheme, it is possible to obtain a closed-form expression for the BER. A closed-form BER expression would serve as an attractive alternative to previously derived bounds for evaluating performance. Moreover, while it is widely understood that Alamouti's scheme has a performance loss of 3 dB with respect to 2-level maximum ratio combining (MRC), to our knowledge this fact has not been shown analytically. Our expressions are derived from the PDF of the phase of the received signal. While the procedure for deriving the BER applies to any M-ary phase shift keying (MPSK) modulation, binary PSK (BPSK) and quadrature PSK (QPSK) examples are worked out in detail. BER expressions are derived for both coherent and differential modulations. For the coherent case, the BER expression is exact, while for the differential case, it is an approximation for high SNR.

II. COHERENT STBC

Assume a communication system with two transmit antennas and one receive antenna. The signal model for the Alamouti's coherent scheme [2] at time slot 1, 2 is given by

$$\begin{bmatrix} r_{1,k} \\ r_{2,k} \end{bmatrix} = \sqrt{E_s} \begin{bmatrix} s_{1,k} & s_{2,k} \\ -s_{2,k}^* & s_{1,k}^* \end{bmatrix} \begin{bmatrix} h_1 \\ h_2 \end{bmatrix} + \begin{bmatrix} n_{1,k} \\ n_{2,k} \end{bmatrix}, \tag{1}$$

where h_1, h_2 are the Gaussian complex fading path gains from transmit antenna 1, 2 to the receive antenna respectively. Path gains are quasi-static, Gaussian random variables with zero-mean and variance $1/2$ per dimension. Noise samples $n_{1,k}, n_{2,k}$ are zero-mean Gaussian random variables with variance $N_0/2$ per dimension; E_s is the total symbol energy, and k denotes the time index. The superscript '$*$' denotes complex conjugation and the symbols $s_{1,k},\ s_{2,k}$ belong to a MPSK constellation with amplitude $1/\sqrt{2}$.

The signal model (1) in vector-matrix form is

$$\mathbf{r}_k = \sqrt{E_s}\mathbf{H}\mathbf{s}_k + \mathbf{n}_k,\ k = 1, 2, ..., L \tag{2}$$

where $\mathbf{r}_k = \left[r_{1,k},\ -r_{2,k}^*\right]^T$; $\mathbf{n}_k$ is defined similarly, the codeword $\mathbf{s}_k = [s_{1,k},\ s_{2,k}]^T$, the superscript '$T$' denotes transposition, and the channel matrix

$$\mathbf{H} = \begin{bmatrix} h_1 & h_2 \\ -h_2^* & h_1^* \end{bmatrix}. \tag{3}$$

Detection is carried out codeword by codeword. For known channel matrix $\mathbf{H}$, the maximum likelihood (ML) detector is given by the maximum of the log-likelihood $L(\widetilde{\mathbf{s}}_k) = \left\|\mathbf{r}_k - \sqrt{E_s}\mathbf{H}\widetilde{\mathbf{s}}_k\right\|^2$, where $\widetilde{\mathbf{s}}_k$ is the tested symbol. Due to the unitary property of $\mathbf{H}$ and the constant power of $\widetilde{\mathbf{s}}_k$ for MPSK, the decision metric $Re\left\{\mathbf{s}_k^\dagger \mathbf{H}^\dagger \mathbf{r}_k\right\}$, where '$\dagger$' denotes the Hermitian operation, is a linear function of the observed data $\mathbf{r}_k$. Let $\boldsymbol{\Upsilon} = [\Upsilon_1\ \ \Upsilon_2]^T = \mathbf{H}^\dagger \mathbf{r}_k$. The term Υ_1 depends only on the transmitted symbol $s_{1,k}$. Indeed,

$$\Upsilon_1 = h_1^*\left(\sqrt{E_s}h_1 s_{1,k} + n_{1,k}\right) + h_2\left(\sqrt{E_s}h_2^* s_{1,k} + n_{2,k}^*\right). \tag{4}$$

Due to symmetry considerations, symbols $s_{1,k}, s_{2,k}$ have the same error probability, hence it suffices to carry out the analysis only for Υ_1. The BER of $s_{1,k}$ can be obtained from the PDF of Υ_1. Define the random variables

$$\begin{aligned} X_1 &= h_1^*, Y_1 = \left(\sqrt{E_s}h_1 s_{1,k} + n_{1,k}\right)^* \\ X_2 &= h_2, Y_2 = \left(\sqrt{E_s}h_2^* s_{1,k} + n_{2,k}^*\right)^*; \end{aligned} \tag{5}$$

then Υ_1 can be expressed as $\Upsilon_1 = \sum_{i=1}^{2} X_i Y_i^*$.

Define $z_r = Re(\Upsilon_1)$ and $z_i = Im(\Upsilon_1)$, then $\Upsilon_1 = z_r + jz_i$. Since the phase of Υ_1 is the decision variable, we set $r = \sqrt{z_r^2 + z_i^2},\ \theta = \tan^{-1}(z_i/z_r)$. According to [7], the probability of θ in an interval $[\theta_1,\ \theta_2]$ is expressed as

$$P(\theta_1 \le \theta \le \theta_2) = -\frac{\left(1-\mu^2\right)^2}{2\pi} \times \frac{\partial}{\partial b}\left[f(b,\alpha_2) - f(b,\alpha_1)\right]\Big|_{b=1}, \tag{6}$$

where the function $f(b,\alpha)$ is defined as follows:

$$f(b,\alpha_i) = \frac{1}{b-\mu^2}\left[\frac{\mu\sqrt{1-(b/\mu^2-1)\alpha_i^2}}{b^{1/2}}\cot^{-1}\alpha_i - \cot^{-1}\left(\frac{\alpha_i b^{1/2}}{\mu\sqrt{1-(b/\mu^2-1)\alpha_i^2}}\right)\right], \tag{7}$$

and

$$\alpha_i = \frac{-\mu\cos\theta_i}{\sqrt{b-\mu^2\cos^2\theta_i}},\ i=1,2. \tag{8}$$

The term μ represents the normalized cross-correlation between X_i and Y_i, for $i=1,2$. To proceed with the BER computation, and without loss of generality, assume symbol $s_{1,k}$ has zero phase, i.e., $s_{1,k}=1/\sqrt{2}$. The normalized cross-correlation is defined as $\mu = m_{xy}/\sqrt{m_{xx}m_{yy}}$, where from (5) $m_{xx}=E\left(|X_i|^2\right)=1$ since by assumption $E\left(|h_i|^2\right)=1$. Also $m_{yy}=E\left(|Y_i|^2\right)=E_s/2+N_0$, since $|s_{1,k}|^2=1/2$. Finally $m_{xy}=E\left(X_iY_i^*\right)=\sqrt{E_s/2}$. It follows that the normalized cross-correlation is given by $\mu=\sqrt{\rho/(\rho+2)}$, where $\rho=E_s/N_0$ is the SNR per symbol.

The BER for MPSK with any number of levels can be evaluated from (6). For BPSK signals, from [7, p. 891], the PDF $p(\theta)$ is an even function. It follows that the BER can be obtained by integrating the PDF $p(\theta)$ over the range $\frac{1}{2}\pi<\theta<\pi$ or equivalently from,

$$P_{2b} = 2P\left(\pi/2\le\theta\le\pi\right). \tag{9}$$

Using (6) in (9) and after some algebraic manipulations, the exact BER for the coherent Alamouti scheme with BPSK is obtained

$$P_{2b} = \frac{1}{2}\left[1-\mu-\frac{1}{2}\mu\left(1-\mu^2\right)\right]. \tag{10}$$

In the QPSK case, a Gray code is used to map pairs of bits into phases. For a transmitted symbol $s_{1,k}$, it is clear that a single bit error is committed when the received phase is $\frac{1}{4}\pi<\theta<\frac{3}{4}\pi$, and a double bit error is committed when the received phase is $\frac{3}{4}\pi<\theta<\pi$. Thus, the BER is expressed as

$$P_{4b} = P\left(\pi/4\le\theta\le 3\pi/4\right)+2P\left(\pi\le\theta\le 3\pi/4\right) \tag{11}$$

Similar to (10), but skipping due to space constraints, the BER for the coherent Alamouti schemes with QPSK modulation is obtained

$$P_{4b} = \frac{1}{2}\left[1-\frac{\mu}{\sqrt{2-\mu^2}}-\frac{\mu\left(1-\mu^2\right)}{\left(2-\mu^2\right)\sqrt{2-\mu^2}}\right]. \tag{12}$$

Expressions (10) and (12) are exact and confirm the conventional wisdom that Alamouti's 2×1 STBC has exactly 3 dB loss at any SNR relative to 1×2 MRC receive diversity [7].

III. Differential STBC

The differential STBC scheme analyzed in this paper is the one recently proposed by Tarokh and Jafarkhani [3] based on the Alamouti transmit diversity scheme [2].

The STBC message $\mathbf{S}_k$ is defined as the unitary orthogonal 2×2 matrix,

$$\mathbf{S}_k = \begin{bmatrix} s_{1,k} & s_{2,k} \\ -s_{2,k}^* & s_{1,k}^* \end{bmatrix}. \tag{13}$$

The message $\mathbf{S}_k$ is differentially encoded similar to the standard single-antenna DPSK. To initialize transmission, the transmitter sends a specified message codeword $\mathbf{C}_0$, in form of a unitary matrix, for example

$$\mathbf{C}_0 = \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ -1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}. \tag{14}$$

The differentially encoded message $\mathbf{C}_k$ at time k is obtained by multiplying the codeword at time $k-1$, $\mathbf{C}_{k-1}$ by the current message $\mathbf{S}_k$, namely $\mathbf{C}_k = \mathbf{S}_k \mathbf{C}_{k-1}$. Consistent with ([3]), this process is initialized with $\mathbf{C}_1 = \mathbf{S}_1 \mathbf{C}_0$. Note that the codeword $\mathbf{C}_k$ has the same unitary property as the message matrix $\mathbf{S}_k$. Obviously, if the codewords $\mathbf{C}_k$ are observable at the receiver, the messages $\mathbf{S}_k$ can be decoded from $\mathbf{C}_k \mathbf{C}_{k-1}^\dagger = \mathbf{S}_k \mathbf{C}_{k-1} \mathbf{C}_{k-1}^\dagger = \mathbf{S}_k$.

Paralleling (1) and using matrix notation, the signal model at the receiver for the two time slots associated with each codeword is expressed

$$\mathbf{R}_k = \sqrt{E_s}\mathbf{C}_k \mathbf{H} + \mathbf{N}_k, \tag{15}$$

where

$$\begin{aligned} \mathbf{R}_k &= \begin{bmatrix} r_{1,k} & -r_{2,k}^* \\ r_{2,k} & r_{1,k}^* \end{bmatrix}, \quad \mathbf{C}_k = \begin{bmatrix} c_{1,k} & c_{2,k} \\ -c_{2,k}^* & c_{1,k}^* \end{bmatrix}, \\ \mathbf{H} &= \begin{bmatrix} h_1 & -h_2^* \\ h_2 & h_1^* \end{bmatrix}, \quad \mathbf{N}_k = \begin{bmatrix} n_{1,k} & -n_{2,k}^* \\ n_{2,k} & n_{1,k}^* \end{bmatrix}. \end{aligned} \tag{16}$$

Matching the differential encoding process, the differential decoder performs the operation $\mathbf{G} = \mathbf{R}_k \mathbf{R}_{k-1}^\dagger / \sqrt{E_s}$. The ML detector for the message $\mathbf{S}_k$ transmitted by the STBC differential scheme employs two consecutive codewords and is given by $\widehat{\mathbf{S}}_k = \arg\max_{s_{1,k}, s_{2,k}} tr \left\{ \mathbf{S}_k^\dagger \mathbf{G} \right\}$, where the operator $tr\{\cdot\}$ denotes the matrix trace.

We wish to express $\mathbf{G}$ in a form convenient to bit error analysis. The error probability of $s_{1,k}$ can be analyzed from the $(1,1)$ term of the 2×2 matrix $\mathbf{G}$. Denote this term as G_1. Then with some simple algebraic manipulations and neglecting the second-order noise term, it is possible to show that G_1 can be expressed as $G_1 \simeq \sum_{k=1}^{2} X_k Y_k^*$, where

$$\begin{aligned} X_1 &= h_1, && (17) \\ Y_1 &= \Big(\sqrt{E_s} h_1^* s_{1,k} + c_{1,k-1} n_{1,k}^* + c_{1,k} n_{1,k-1}^* \\ &\quad - c_{2,k-1}^* n_{2,k}^* - c_{2,k}^* n_{2,k-1}^* \Big)^*, && (18) \\ X_2 &= h_2^*, && (19) \\ Y_2 &= \Big(\sqrt{E_s} h_2 s_{1,k} + c_{2,k-1}^* n_{1,k} + c_{1,k-1} n_{2,k} \\ &\quad + c_{2,k}^* n_{1,k-1} + c_{1,k} n_{2,k-1} \Big)^*. && (20) \end{aligned}$$

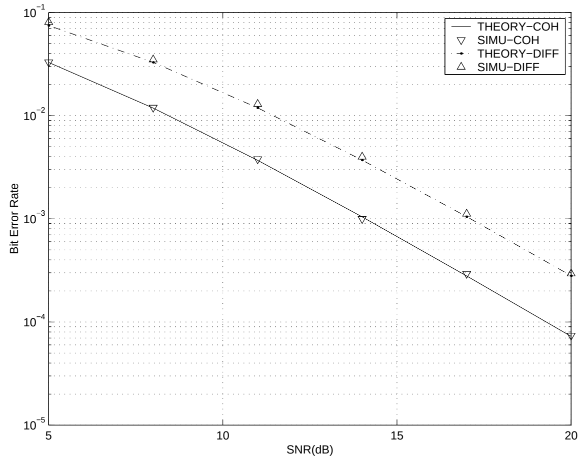

Fig. 1. Comparison of analysis and simulation with coherent and differential detection in BPSK case (2T1R antennas, 1 bit/s/Hz)

These approximations are suitable for high SNR such that second-order noise terms are negligible. Following the same steps as in Section II, BER expressions are obtained for BPSK and QPSK modulations. These expressions are identical to (10) and (12) respectively, except the SNR per symbol for the differential case is $\rho_d = E_s/2N_0$ (3 dB loss compared to coherent detection).

IV. Numerical Results

Numerical results are provided to demonstrate the analysis developed in this letter and to compare it with simulation results. Fig. 1 shows the BER versus the SNR for binary coherent and differential STBC. Curves were obtained both by analysis and by simulations as indicated by the figure annotations. A very good match is observed between the analysis and simulation. The slight bias at low SNR for the differential case is attributed to the second order noise terms, which were neglected in the analysis. Fig. 2 presents the case of coherent and differential STBC with QPSK modulation. The figures also confirm that an approximately 3 dB performance gap exists between the coherent and differential schemes. Note that our performance for differential STBC with QPSK is a little better than [3]'s since bits are mapped to QPSK symbols using the Gray code.

V. Conclusions

We derived closed-form expressions of the BER over slow, flat Rayleigh fading for coherent and differential schemes based on Alamouti's STBC. While the procedure outlined is applicable to any MPSK modulation, explicit BER expressions were obtained for BPSK and QPSK. Comparison of analytical and simulation results validates the new expressions. The closed-form expressions show that approximately 3 dB SNR loss is incurred by the differential scheme compared to the coherent case. The method presented is extendable to other MIMO channels.

References

[1] V. Tarokh and A. Seshadri, "Space-time codes for high data rate wireless communication: performance criterion and code construction," *IEEE Transactions on Information Theory*, March, 1998.

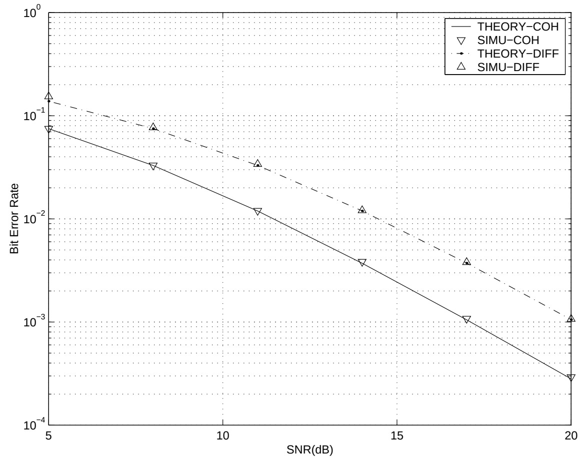

Fig. 2. Comparison of analysis and simulation with coherent and non-coherent detection in QPSK case (2T1R antennas, 2 bits/s/Hz)

[2] S. Alamouti, “A simple transmitter diversity scheme for wireless communications,” *IEEE Journal on Selected Areas in Communications*, vol. 16, pp. 1451 –1458, Oct. 1998.

[3] V. Tarokh and H. Jafarkhani, “A differential detection scheme for transmit diversity,” *IEEE Journal on Selected Areas in Communications*, vol. 18, pp. 1169 –1174, Jul. 2000.

[4] B. Hochwald and W. Sweldens, “Differential unitary space-time modulation,” *IEEE Transactions on Communications*, vol. 48, pp. 2041 –2052, Dec. 2000.

[5] B. Hughes, “Differential space-time modulation,” *IEEE Transactions on Information Theory*, vol. 46, pp. 2567–2578, Nov. 2000.

[6] X. Li, T. Luo, G. Yue, and C. Yin, “A squaring method to simplify the decoding of orthogonal space-time block codes,” *IEEE Transactions on Communications*, vol. 49, pp. 1700 –1703, Oct. 2001.

[7] J. G. Proakis , *Digital Communications, Third Edition, McGraw-Hill, Inc.*, 1998.

[8] G. Bauch and J. Hagenauer, “Analytical evaluation of space-time transmit diversity with FEC-coding,” *IEEE GLOBECOM ’01*, vol. 1, pp. 435 –439, 2001.