# Exact Closed-Form Performance Analysis of Optimum Combining With Multiple Cochannel Interferers and Rayleigh Fading

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***Abstract*—A new closed-form expression is derived for the exact bit-error probability (BEP) for optimum combining with binary phase-shift keying. The exact BEP expression is for multiple, equal power, cochannel interferers and multiple reception branches. It is assumed that the aggregate interference and noise is Gaussian and that both the desired signal and interference are subject to Rayleigh fading. The derivation starts by expressing the optimum combining decision statistic as a sum of quadratic forms of Gaussian random variables and it proceeds to average over the fading interference. The new BEP expression has low complexity as it contains only finite sums and products.**

***Index Terms*—Diversity reception, error probability performance, fading channels, interference suppression, optimum combining.**

## I. Introduction

OPTIMUM combining is a well-known method to combat fading and suppress cochannel interference in wireless communication systems with reception diversity. It combines the outputs of the reception branches in an optimum way and achieves the maximum output signal-to-interference-plus-noise ratio (SINR).

Performance analysis of optimum combining has been an active research area. Analysis for the case of a single interference source can be found in [1]–[3]. In [1] and [2], Rayleigh fading is assumed for the desired signal, but mean values, rather than actual distributions, are used to represent fading effects on the interference. In [3], exact expressions (requiring integration) are developed under the assumption of Rayleigh fading for both the desired signal and interference. Closed-form expressions of the bit-error probability (BEP) for this case were obtained in [4].

The case of multiple interferers is more challenging. Closed-form expressions of the BEP for the number of interferers no less than the number of reception branches and negligible thermal noise with binary phase-shift keying (BPSK) modulation were developed in [5]. To our knowledge, no exact, closed-form expressions have been published for Rayleigh fading diversity channels, BPSK or $M$-ary phase-shift keying (MPSK) with multiple interferers and nonnegligible thermal noise. The performance of such systems has been studied extensively through the use of Monte Carlo simulations [2], capacity [6], upper bound [7], approximate expressions [8], and exact expressions with integral forms ([9], [10]). A summary of recent results on optimum combining can be found in [11].

The conventional way of deriving the expression for symbol-error probability (SEP) or BEP starts with deriving the probability density function (PDF) of the SINR conditioned on channel realizations of the desired signal and the interference. For an exact conditional SEP, the Gaussian assumption is necessary for the interference plus noise. The unconditional SEP is obtained by averaging first over the desired signal channel and then averaging over the interference channel.

In this paper, we take a different approach by performing the analysis directly on the decision statistic rather than on the SINR. We show that for BPSK, this approach allows exact BEP analysis and it requires only averaging over the fading of the interference. Although the algebra is somewhat cumbersome, at the end this method provides a closed-form expression.

The paper is organized as follows. Following the system model in Section II, we derive the conditional BEP in Section III. In Section IV, we average the conditional BEP over the fading of the interference to get the unconditional BEP. Numerical results are presented in Section V, and finally, conclusions are drawn in Section VI.

## II. System Model

Consider a wireless communication system with $N$ independent reception branches and $L+1$ users. Of the users, one is the desired user and it transmits signals with power $P_s$. The other $L$ sources are considered interference. Assuming perfect carrier demodulation and synchronization, the sampled output of the matched filter for the $l$th branch is

$$r_l = \sqrt{P_s} c_l s + \sum_{i=1}^{L} \sqrt{P_I} c_{i,l} s_i + n_l, \qquad l = 1, 2, \ldots, N \quad (1)$$

where $c_l$ and $s$ are, respectively, the channel gain and BPSK symbol of the desired signal; $c_{i,l}$ and $s_i$ are, respectively, the $i$th interferer's channel gain and symbol; and $P_I$ is the interference power (assumed equal for all interference sources). The term $n_l$ represents additive white Gaussian noise (AWGN). The channel gains $c_l$ and $c_{i,l}$ are assumed to be independently and identically distributed (i.i.d.) zero-mean, circularly symmetric,

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complex Gaussian random variables (Rayleigh fading), with variance 1/2 per dimension. The signal model in vector notation is

$$\mathbf{r} = \sqrt{P_s}\mathbf{c}s + \sqrt{P_I}\sum_{i=1}^{L}\mathbf{c}_i s_i + \mathbf{n} = \sqrt{P_s}\mathbf{c}s + \mathbf{z} \tag{2}$$

where $\mathbf{r} = [r_1, r_2, \ldots, r_N]^T$, $\mathbf{c}$, $\mathbf{c}_i$, and $\mathbf{n}$ are defined similarly, $\mathbf{z} = \sqrt{P_I}\sum_{i=1}^{L}\mathbf{c}_i s_i + \mathbf{n}$ is the interference plus noise vector, and the superscript $T$ denotes vector transposition.

It is further assumed that conditioned on the vectors $\mathbf{c}_i$, the interference plus noise vector $\mathbf{z}$ has a multivariate complex-Gaussian distribution with zero mean and covariance matrix $\mathbf{R} = E[\mathbf{z}\mathbf{z}^H]$

$$\mathbf{R} = P_I\sum_{i=1}^{L}\mathbf{c}_i\mathbf{c}_i^H + \sigma^2\mathbf{I}_N \tag{3}$$

where the superscript $H$ denotes the Hermitian transposition, $\sigma^2$ is the power of the noise, and $\mathbf{I}_N$ is an identity matrix of rank $N$.

Define $N_{\max} = \max(N, L)$ and $N_{\min} = \min(N, L)$. Diagonalize $\mathbf{R}$ as $\mathbf{R} = \mathbf{U}\Lambda\mathbf{U}^H$, where $\Lambda = \mathrm{diag}\,(\lambda_1, \lambda_2, \ldots, \lambda_N)$, $\lambda_1, \lambda_2, \ldots, \lambda_N$ are the eigenvalues of $\mathbf{R}$, and $\mathbf{U}$ is the unitary matrix whose columns are the eigenvectors of $\mathbf{R}$. Assume that the vectors $\mathbf{c}_i$ (for $i = 1, 2, \ldots, L$) are linearly independent (a reasonable assumption, since the components of these vectors are realizations of mutually independent random variables), and adopt a convention whereby eigenvalues are listed in descending order. It follows that [10] $\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_{N_{\min}}$ are random variables, while $\lambda_m = \sigma^2$ for $m = N_{\min}+1, N_{\min}+2, \ldots, N$. For later use, we denote the set of nontrivial eigenvalues as $\lambda = \{\lambda_1, \lambda_2, \ldots, \lambda_{N_{\min}}\}$. The inverse covariance matrix of $\mathbf{R}$ is $\mathbf{R}^{-1} = \mathbf{U}\Lambda^{-1}\mathbf{U}^H$.

## III. Derivation of Conditional BEP

In this and the next section, we carry out the theoretical analysis of the BEP of optimum combining for BPSK modulation in the presence of multiple interference sources when both the desired signal and interference are subject to Rayleigh fading.

With the optimum combining detector, the received signal vector $\mathbf{r}$ is weighted and combined to obtain the output signal. The weight vector that yields the maximum SINR is $\mathbf{w} = \mathbf{R}^{-1}\mathbf{c}$ [2].[1] The output of the combiner is $\mathbf{w}^H\mathbf{r}$. For BPSK modulation, the decision rule of the detector is, if $\Re(\mathbf{w}^H\mathbf{r}) \geq 0$ (where $\Re(\cdot)$ denotes the real-part operation), the decision is made that 1 is transmitted; otherwise, the decision is made that $-1$ is transmitted. Due to the symmetry of the BPSK constellation and assuming a source with equal symbol probabilities, it suffices to analyze the case of $s = 1$. For this case, the received signal is $\mathbf{r} = \sqrt{P_s}\mathbf{c} + \mathbf{z}$. Define

$$D = 2\Re(\mathbf{w}^H\mathbf{r}) = \mathbf{w}^H\mathbf{r} + (\mathbf{w}^H\mathbf{r})^*. \tag{4}$$

[1] In this paper, $R$ is the interference-plus-noise covariance matrix. Some authors compute the optimal combining weight vector from the signal-plus-interference-and-noise covariance matrix. As shown in [12] and can be readily verified, the resulting weight vectors provide the same performance as they differ only by a scaling factor.

According to the decision rule, when $D < 0$, the decision is made that $-1$ is transmitted and an error occurs. Therefore, the BEP is $P_e = \Pr(D < 0)$. The analysis has two steps. First, the BEP is expressed conditioned on the fading of the interference. Subsequently, the conditioned BEP is averaged over the fading of the interference.

Fixing the values of the channels $\mathbf{c}_i$ of the interference sources, leads to fixed values of eigenvalues of interference-plus-noise covariance matrix $\mathbf{R}$. These eigenvalues $\lambda_m$ form the diagonal of the matrix $\Lambda$. Substituting $\mathbf{w} = \mathbf{R}^{-1}\mathbf{c}$ into (4) and decomposing $\mathbf{R}^{-1}$ as $\mathbf{U}\Lambda^{-1}\mathbf{U}^H$, we can express $D$ as

$$D = \sum_{m=1}^{N}\lambda_m^{-1}\left(g_m^* x_m + g_m x_m^*\right) \tag{5}$$

where $\lambda_m$'s are the eigenvalues of $\mathbf{R}$ defined previously, $x_m$'s are elements of the whitened observation vector $\mathbf{x} = [x_1, x_2, \ldots, x_N]^T = \mathbf{U}^H\mathbf{r}$, and $g_m$'s are elements of the whitened channel vector $\mathbf{g} = [g_1, g_2, \ldots, g_N]^T = \mathbf{U}^H\mathbf{c}$. Conditioned on the eigenvalues $\lambda_m$, the variable $D$ is a quadratic form of Gaussian random variables. Our goal is to evaluate the conditional BEP $P(e|\lambda) = \Pr(D < 0|\lambda)$, where the notation indicates the dependency on the $N_{\min}$ largest eigenvalues of $\mathbf{R}$ (the other $(N - N_{\min})$ eigenvalues are equal to constant $\sigma^2$).

Let $\Phi_D(j\omega)$ be the characteristic function of $D$ conditioned on $\lambda$. Using results from [13, App. B], it can be shown that the conditional BEP is

$$\begin{aligned} P(e|\lambda) &= -\frac{1}{2\pi j}\int_{-\infty+j\varepsilon}^{\infty+j\varepsilon}\frac{\Phi_D(j\omega)}{\omega}d\omega \\ &= -\sum_{\mathrm{Im}(\omega_n)>0}\mathrm{Res}\left[\frac{\Phi_D(j\omega)}{\omega};\omega_n\right] \end{aligned} \tag{6}$$

where $\varepsilon$ is a small positive number and $\mathrm{Res}[\Phi_D(j\omega)/\omega;\omega_n]$ denotes the residue of $\Phi_D(j\omega)/\omega$ at pole $\omega_n$. The summation is taken over the poles in the upper half of the complex plane. According to the complex variables theory [14], if a function $f(\omega)$ has a pole $\omega_0$ of order $m$, the residue of $f(\omega)$ at $\omega_0$ is

$$\mathrm{Res}\left[f(\omega);\omega_0\right] = \frac{1}{(m-1)!}\frac{d^{(m-1)}}{d\omega^{(m-1)}}\left[(\omega-\omega_0)^m f(\omega)\right]\Bigg|_{\omega=\omega_0}. \tag{7}$$

In Appendix A, it is shown that the characteristic function $\Phi_D(j\omega)$ is expressed as shown in (8) at the bottom of the next page.

Substituting (8) into (6), and carrying out the calculation of the residues, we obtain the conditional BEP as shown in (10) at the bottom of the next page. When the number of reception branches $N$ is less than or equal to the number of interferers, i.e., $N \leq L$, $N_{\min} = N$, the summation $\sum_{l=0}^{N-N_{\min}-1}$ in (9) is equal to zero.

As an example, we derive the expression of BEP for the special case of no interference.

### A. Special Case: No Interference

In this case, $N_{\min} = L = 0$, optimum combining becomes maximum ratio combining. Equation (8) becomes

$$\Phi_D(j\omega) = \left\{ \frac{\sigma^2}{\left[\omega + j(\sqrt{P_s+\sigma^2} - \sqrt{P_s})\right]\left[\omega - j(\sqrt{P_s+\sigma^2} + \sqrt{P_s})\right]} \right\}^N. \tag{11}$$

Substituting the above expression in (6), and carrying out the calculation of the residue, we get

$$P_e = \left[\frac{1}{2}(1-\mu)\right]^N \left\{ \sum_{k=0}^{N-1} \binom{N-1+k}{k} \left[\frac{1}{2}(1+\mu)\right]^k \right\} \tag{12}$$

where $\mu = \sqrt{\gamma/(1+\gamma)}$, and $\gamma = P_s/\sigma^2$ is the signal-to-noise ratio (SNR). (12) is the same as (14-4-15) in [13]. Note that the derivation of (12) did not require integration. In [13], the BEP of maximum ratio combining is obtained by integration. It was shown that [13, eq. (14-1-18)] for SNR $\gamma \gg 1$

$$P_e \approx \binom{2N-1}{N} \frac{1}{4^N \gamma^N}. \tag{13}$$

Equation (13) will be used for comparison later.

## IV. Derivation of Unconditional BEP

For the general case with interference, the unconditional BEP $P_e$ is obtained by averaging the conditional BEP $P(e|\lambda)$ over the fading of the interference $\mathbf{c}_i$, or equivalently, over the eigenvalues $\lambda = \{\lambda_1, \lambda_2, \ldots, \lambda_{N_{\min}}\}$

$$P_e = \int \ldots \int P(e|\lambda) p_\lambda(\lambda) d\lambda \tag{14}$$

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$$\Phi_D(j\omega) = \left\{ \frac{\sigma^2}{\left[\omega + j(\sqrt{P_s+\sigma^2} - \sqrt{P_s})\right]\left[\omega - j(\sqrt{P_s+\sigma^2} + \sqrt{P_s})\right]} \right\}^{N-N_{\min}} \prod_{m=1}^{N_{\min}} \frac{\lambda_m}{\left[\omega + j(\sqrt{P_s+\lambda_m} - \sqrt{P_s})\right]\left[\omega - j(\sqrt{P_s+\lambda_m} + \sqrt{P_s})\right]}. \tag{8}$$

$$\begin{aligned} P(e|\lambda) = {} & (-1)^{N+1} \sum_{m=1}^{N_{\min}} \frac{\lambda_m}{2\sqrt{P_s+\lambda_m}(\sqrt{P_s+\lambda_m}+\sqrt{P_s})} \\ & \times \frac{(\sigma^2)^{N-N_{\min}}}{(\lambda_m-\sigma^2)^{N-N_{\min}}} \prod_{n=1}^{m-1} \frac{\lambda_n}{(\lambda_m-\lambda_n)} \\ & \times \prod_{n=m+1}^{N_{\min}} \frac{\lambda_n}{(\lambda_m-\lambda_n)} + (\sigma^2)^{N-N_{\min}} \\ & \times \sum_{l=0}^{N-N_{\min}-1} \binom{N-N_{\min}+l-1}{l} \\ & \left\{ 1 + \frac{1}{2}(-1)^{N_{\min}} \left(\sqrt{P_s+\sigma^2}+\sqrt{P_s}\right)^{N-N_{\min}-l} \right. \\ & \times \left(\prod_{n=1}^{N_{\min}} \lambda_n\right) \sum_{m=1}^{N_{\min}} \left(\prod_{n=1}^{m-1} \frac{1}{\lambda_m-\lambda_n}\right) \\ & \times \left(\prod_{n=m+1}^{N_{\min}} \frac{1}{\lambda_m-\lambda_n}\right) \frac{1}{\sqrt{P_s+\lambda_m}} \\ & \times \left[ \frac{1}{(\sqrt{P_s+\lambda_m}-\sqrt{P_s})} \frac{1}{(\sqrt{P_s+\sigma^2}+\sqrt{P_s+\lambda_m})^{N-N_{\min}-l}} \right. \\ & \left.\left. + \frac{1}{(\sqrt{P_s+\lambda_m}+\sqrt{P_s})} \frac{1}{(\sqrt{P_s+\sigma^2}-\sqrt{P_s+\lambda_m})^{N-N_{\min}-l}} \right] \right\} \end{aligned} \tag{9}$$

$$\times \frac{1}{(\sqrt{P_s+\sigma^2}+\sqrt{P_s})^{N-N_{\min}-l}(2\sqrt{P_s+\sigma^2})^{N-N_{\min}+l}} \tag{10}$$

where $p_\lambda(\lambda)$ is the joint PDF of the eigenvalues. The PDF $p_\lambda(\lambda)$ was developed in [10] for a signal model similar to ours and is given by

$$p_\lambda(\lambda) = K_0 \frac{1}{P_I^{N_{\min}}} \left[ \prod_{i=1}^{N_{\min}} \exp\left(-\frac{\lambda_i - \sigma^2}{P_I}\right) \times \left(\frac{\lambda_i - \sigma^2}{P_I}\right)^{N_{\max}-N_{\min}} \right] \times \left[ \prod_{1\le i<j\le N_{\min}} \left(\frac{\lambda_i - \sigma^2}{P_I} - \frac{\lambda_j - \sigma^2}{P_I}\right)^2 \right] \quad (15)$$

where

$$K_0 = \frac{1}{\left[\prod_{i=1}^{N_{\min}} (N_{\max} - i)!\right] \left[\prod_{i=1}^{N_{\min}} (N_{\min} - i)!\right]}. \quad (16)$$

The conditional BEP in (10) is a nonrational function of the eigenvalues $\lambda_m$'s. To facilitate the integration in (14), we define the following transformation of variables:

$$y_m = \sqrt{\frac{\lambda_m}{P_s} + 1}, \qquad m = 1, 2, \ldots, N_{\min} \quad (17)$$

and define the set $\mathbf{y} = \{y_1, y_2, \ldots, y_{N_{\min}}\}$. Since $\lambda_m$ is random, $y_m$ is random as well. Also define

$$\eta = \sqrt{\frac{\sigma^2}{P_s} + 1} = \sqrt{\frac{1}{\gamma} + 1}. \quad (18)$$

Then we have

$$\lambda_m = P_s\left(y_m^2 - 1\right), \qquad m = 1, 2, \ldots, N_{\min} \quad (19)$$

$$\sigma^2 = P_s(\eta^2 - 1). \quad (20)$$

Substituting (19) and (20) into (10), and after some straightforward manipulations, we get the conditional BEP as a function of the variables $y_m$'s

$$P(e|\mathbf{y}) = -\sum_{m=1}^{N_{\min}} f_m(\mathbf{y}) + (-1)^{N-N_{\min}} \times \sum_{l=0}^{N-N_{\min}-1} \binom{N-N_{\min}+l-1}{l} \times \left[1 + \sum_{m=1}^{N_{\min}} h_{m,l}(\mathbf{y})\right] \frac{(1-\eta)^{N-N_{\min}-l}}{(2\eta)^{N-N_{\min}+l}} \left(-\frac{1}{\gamma}\right)^l \quad (21)$$

where the functions $f_m(\mathbf{y})$ and $h_{m,l}(\mathbf{y})$ are defined, respectively, as

$$f_m(\mathbf{y}) = \frac{1-y_m}{2y_m} \frac{(1-\eta^2)^{N-N_{\min}}}{(y_m^2-\eta^2)^{N-N_{\min}}} \left\{\prod_{n=1}^{m-1} \frac{1-y_n^2}{y_m^2-y_n^2}\right\} \times \left\{\prod_{n=m+1}^{N_{\min}} \frac{1-y_n^2}{y_m^2-y_n^2}\right\} \quad (22)$$

and

$$h_{m,l}(\mathbf{y}) = (-1)^{N-N_{\min}-l} \times \frac{(1+\eta)^{N-N_{\min}-l}}{2y_m} \frac{1}{(y_m^2-\eta^2)^{N-N_{\min}-l}} \times b_{N-N_{\min}-l}(y_m) \left\{\prod_{n=1}^{m-1} \frac{1-y_n^2}{y_m^2-y_n^2}\right\} \times \left\{\prod_{n=m+1}^{N_{\min}} \frac{1-y_n^2}{y_m^2-y_n^2}\right\}. \quad (23)$$

The function $b_k(y_m)$ in (23) is, in turn, defined for $1 \le k \le N - N_{\min}$ as

$$b_k(y_m) = -(1+y_m)(\eta - y_m)^k + (1-y_m)(\eta + y_m)^k. \quad (24)$$

Clearly, the conditional BEP $P(e|\mathbf{y})$ is a rational function of the elements of the set $\mathbf{y}$. Using the Jacobian of the transformation from $\lambda$ to $\mathbf{y}$, we get the joint PDF of $\mathbf{y}$ as

$$p_{\mathbf{y}}(\mathbf{y}) = K_1 \left\{\prod_{i=1}^{N_{\min}} \exp\left[-\beta\left(y_i^2 - \eta^2\right)\right] \left(y_i^2 - \eta^2\right)^{N_{\max}-N_{\min}}\right\} \times \left[\prod_{1\le i<j\le N_{\min}} \left(y_i^2 - y_j^2\right)^2\right] y_1, y_2, \ldots, y_{N_{\min}} \quad (25)$$

for $y_1 \geqslant y_2 \geqslant \ldots \geqslant y_{N_{\min}} \geqslant \eta$, where $\beta = P_s/P_I$ is the signal-to-interference ratio (SIR), and

$$K_1 = \frac{2^{N_{\min}}}{\left[\prod_{i=1}^{N_{\min}} (N_{\max} - i)!\right] \left[\prod_{i=1}^{N_{\min}} (N_{\min} - i)!\right]} \beta^{N_{\max} N_{\min}}. \quad (26)$$

The unconditional BEP $P_e$ is obtained by averaging the conditional $P(e|\mathbf{y})$ over the random variables in the set $\mathbf{y}$

$$\begin{aligned} P_e &= \int \ldots \int P(e|\mathbf{y}) p_{\mathbf{y}}(\mathbf{y}) d\mathbf{y} \\ &= -\sum_{m=1}^{N_{\min}} \int \ldots \int f_m(\mathbf{y}) p_{\mathbf{y}}(\mathbf{y}) d\mathbf{y} + (-1)^{N-N_{\min}} \\ &\quad \times \sum_{l=0}^{N-N_{\min}-1} \binom{N-N_{\min}+l-1}{l} \left(-\frac{1}{\gamma}\right)^l \\ &\quad \times \left[1 + \sum_{m=1}^{N_{\min}} \int \ldots \int h_{m,l}(\mathbf{y}) p_{\mathbf{y}}(\mathbf{y}) d\mathbf{y}\right] \\ &\quad \times \frac{(1-\eta)^{N-N_{\min}-l}}{(2\eta)^{N-N_{\min}+l}}. \end{aligned} \quad (27)$$

This expression can be used for any number of diversity branches $N$ and any number of interferers $L$. When $N \le L$, $N_{\min} = N$, the summation $\sum_{l=0}^{N-N_{\min}-1}$ is equal to zero, therefore only the first term $-\sum_{m=1}^{N_{\min}} \int \ldots \int f_m(\mathbf{y}) p_{\mathbf{y}}(\mathbf{y}) d\mathbf{y}$ is required to calculated the BEP for $N \le L$.

We proceed to evaluate the terms of (27).

### A. Evaluation of $\sum_{m=1}^{N_{min}} \int \ldots \int f_m(\mathbf{y}) p_{\mathbf{y}}(\mathbf{y}) d\mathbf{y}$

The following definitions are needed.

1) $B_q$ is a sequence defined as

$$B_q = \begin{cases} \sqrt{\frac{\pi}{\beta}} \exp(\beta\eta^2) Q(\sqrt{2\beta}\eta), & q = 0 \\ \frac{\eta}{2\beta} + \left(\frac{1}{2\beta} - \eta^2\right) \sqrt{\frac{\pi}{\beta}} \exp(\beta\eta^2) Q(\sqrt{2\beta}\eta), & q = 1 \\ \left(\frac{2q-1}{2\beta} - \eta^2\right) B_{q-1} + \frac{q-1}{\beta} \eta^2 B_{q-2}, & q \ge 2. \end{cases} \quad (28)$$

The values of $B_q$ for $q \geq 2$ can be evaluated recursively from the values of $B_{q-1}$ and $B_{q-2}$. Since (28) is a second-order difference equation with initial values for $B_q$, by using the method detailed in [15], we can solve this equation and express $B_q$ (for $q \geq 2$) in terms of the initial values $B_0$ and $B_1$ as

$$B_q = v_{q-1,1}B_1 + v_{q-1,2}B_0 \tag{29}$$

where

$$v_{q,j} = \sum_{r=1}^{q+j-1} \sum_{\substack{l_1+\ldots+l_r=q+j-1 \\ l_i\in\{1,2\}, l_r\geq j}} \left[\prod_{m=1}^{r} e_{q-\sum_{n=1}^{m-1} l_n, l_m}\right] \tag{30}$$

and $e_{q,1} = (2q+1)/(2\beta) - \eta^2$, $e_{q,2} = q\eta^2/\beta$. The second summation in (30) is taken over all sets of indexes satisfying the stated conditions. Substituting $B_0$ and $B_1$ in (29), we have $B_q$ (for $q \geq 2$) in closed form

$$B_q = v_{q-1,1}\frac{\eta}{2\beta} + \left[\left(\frac{1}{2\beta} - \eta^2\right) v_{q-1,1} + v_{q-1,2}\right] \times \sqrt{\frac{\pi}{\beta}} \exp(\beta\eta^2) Q(\sqrt{2\beta}\eta). \tag{31}$$

2) $H_{p,q}$ is a function of integers $p$ and $q$. For $0 \leq p, q \leq N_{\min} - 1$

$$H_{p,q} = \frac{1}{\left[\prod_{i=1}^{N_{\min}} (N_{\max} - i)!\right] \left[\prod_{i=1}^{N_{\min}} (N_{\min} - i)!\right]} \times \sum_{\substack{m_1+\ldots+m_{N_{\min}-1}=N_{\min}-1-p \\ m_i\in\{0,1\}}} \times \sum_{\substack{n_1+\ldots+n_{N_{\min}-1}=N_{\min}-1-q \\ n_i\in\{0,1\}}} \det \mathbf{W} \tag{32}$$

where for $N_{\min} = 1$, $\det \mathbf{W} = 1$; for $N_{\min} > 1$, $\det \mathbf{W}$ is the determinant of an $(N_{\min} - 1) \times (N_{\min} - 1)$ matrix whose $i$th row, $j$th column element is

$$W_{i,j} = (m_j + n_j + N_{\max} - N_{\min} + i + j - 2)!.$$

Using these definitions, in Appendix B it is shown that

$$\sum_{m=1}^{N_{\min}} \int \cdots \int f_m(\mathbf{y}) p_{\mathbf{y}}(\mathbf{y}) d_{\mathbf{y}} = \left(-\frac{1}{\gamma}\right)^{N-N_{\min}} \beta^{N_{\max}-N_{\min}+1} \times \sum_{p=0}^{N_{\min}-1} \sum_{q=0}^{N_{\min}-1} (-1)^q H_{p,q} \times \frac{1}{\gamma_p} \left(B_{N_{\max}-N+q} - \frac{1}{2} \frac{(N_{\max}-N+q)!}{\beta^{N_{\max}-N+q+1}}\right) \times \beta^{p+q}. \tag{33}$$

*B. Evaluation of* $\sum_{m=1}^{N_{min}} \int \ldots \int h_{m,l}(\mathbf{y}) p_{\mathbf{y}}(\mathbf{y}) d\mathbf{y}$

By mathematical induction, it can be proved that the function $b_k(y_m)$ defined in (24) can be alternatively expressed as

$$b_k(y_m) = 2y_m \sum_{t=0}^{\left[\frac{k}{2}\right]} a_{k,t} \left(y_m^2 - \eta^2\right)^t \tag{34}$$

where $[k/2]$ denotes the largest integer that is equal to or less than $k/2$, and $a_{k,t}$ is evaluated as

$$a_{k,t} = \left[\binom{k-1-t}{t}(1-\eta) - 2\eta\binom{k-1-t}{t-1}\right](2\eta)^{k-1-2t}. \tag{35}$$

When calculating $a_{k,t}$, we assume $\binom{m}{n} = 0$ for $m < n$ or $n < 0$.

Substituting (34) in (23) and using steps similar to those in Appendix B, we have

$$\begin{aligned}\sum_{m=1}^{N_{\min}} \int \cdots \int h_{m,l}(\mathbf{y}) p_{\mathbf{y}}(\mathbf{y}) d\mathbf{y} &= (-1)^{N-N_{\min}-l} (1+\eta)^{N-N_{\min}-l} \beta^{N-N_{\min}+1} \\ &\times \sum_{t=0}^{\left[\frac{(N-N_{\min}-l)}{2}\right]} a_{N-N_{\min}-l,t} \frac{1}{\beta^{l+t+1}} \sum_{p=0}^{N_{\min}-1} \\ &\times \sum_{q=0}^{N_{\min}-1} (-\mathbf{1})^q (N_{\max} - N + l + t + q)! \\ &\times H_{p,q} \left(\frac{\beta}{\gamma}\right)^p. \end{aligned} \tag{36}$$

Using the expressions obtained in (33) and (36), we can proceed to evaluate (27) to obtain the exact BEP for any given number of diversity branches $N$, number of interferers $L$, SNR $\gamma = P_s/\sigma^2$, and SIR $\beta = P_s/P_I$.

A simpler expression can be derived for the special case of $N > L$, SNR $\gamma \gg 1$, and SIR $\beta \ll 1$.

*Special Case: $N>L$, SNR $\gamma\gg1$, and SIR $\beta\ll1$:* In this case, $N_{\min} = L$. Both (33) and (36) contain the term $\beta^{N-N_{\min}+1}$. Since $\beta \ll 1$ and $N > N_{\min}$, terms containing $\beta^{N-N_{\min}+1}$ are neglected. We have $\sum_{m=1}^{N_{\min}} \int \ldots \int f_m(\mathbf{y}) p_{\mathbf{y}}(\mathbf{y}) d\mathbf{y} \approx 0$ and $\sum_{m=1}^{N_{\min}} \int \ldots \int h_{m,l}(\mathbf{y}) p_{\mathbf{y}}(\mathbf{y}) d\mathbf{y} \approx 0$. Substituting these approximations in (27) and using $\eta = \sqrt{1/\gamma + 1} \approx 1 + 1/(2\gamma)$ for $\gamma \gg 1$, we obtain

$$P_e \approx \binom{2(N-L)-1}{(N-L)} \frac{1}{4^{N-L}\gamma^{N-L}}. \tag{37}$$

Comparing (37) with (13), we can see that for SNR $\gamma \gg 1$, the BEP of a system with $N$ diversity branches and $L(L < N)$ large interferers is equivalent to that of a system with $(N - L)$ diversity branches but without interference. This is a well-known result for optimum combining [6].

## V. Numerical Results

As mentioned in the introduction, the performance of optimum combining has been evaluated in several publications (without the benefit of exact closed-form expressions). Our focus in this paper is not the performance of optimum combining, rather, we use Monte Carlo simulations to demonstrate the new exact BEP expression.

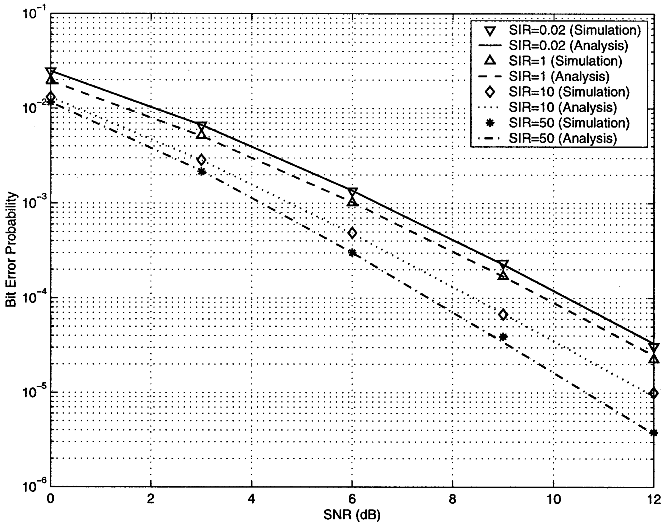


Fig. 1. BEP versus SNR for $N = 4$ branches, $L = 1$ Gaussian distributed interferer.

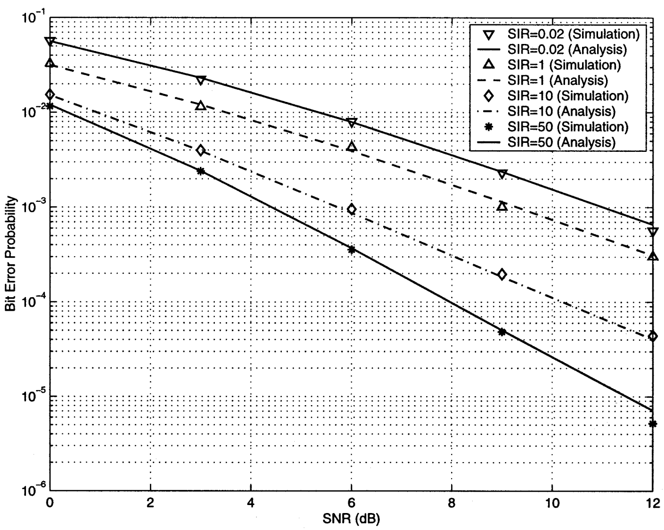


Fig. 2. BEP versus SNR for $N = 4$ branches, $L = 2$ BPSK interferers.

Figs. 1–4 show the BEP versus SNR for different SIR $\beta$. Figs. 1–3 are for $N = 4$ diversity branches, and $L$ =1, 2, 3 interferers, respectively. Fig. 4 is for $N = 8$ diversity branches and $L = 5$ interferers. Fig. 5 is for four reception branches, varying number of interferers, and SIR = 10.

In Figs. 1, 3, 4, and 5, the interference generated in the simulations had a Gaussian distribution as assumed in developing the BEP analysis. Simulation results in Fig. 2 were generated for two interference sources transmitting BPSK symbols. Analytical results were calculated using (27) and related expressions such as (33) and (36).

In all the figures, the analysis results match the simulation results. This provides convincing demonstration of the validity of the analytical expression for BEP.

As shown by Fig. 2 for BPSK interference, the Gaussian assumption for the interference, while necessary for obtaining the theoretical results, is not critical for the accuracy of the BEP expressions. This can be explained by recognizing that the system has a sufficient number of degrees of freedom to suppress the

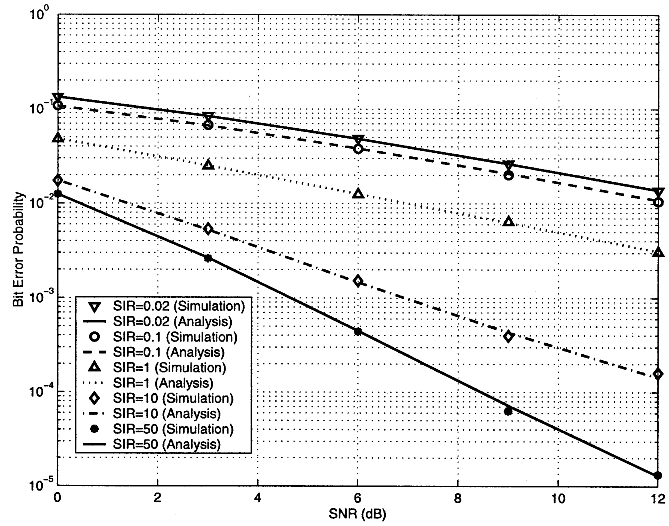


Fig. 3. BEP versus SNR for $N = 4$ branches, $L = 3$ Gaussian distributed interferers.

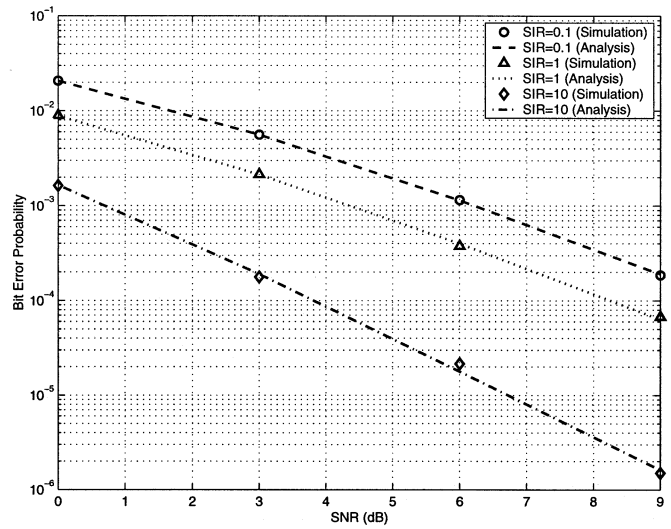


Fig. 4. BEP versus SNR for $N = 8$ branches, $L = 5$ Gaussian distributed interferers.

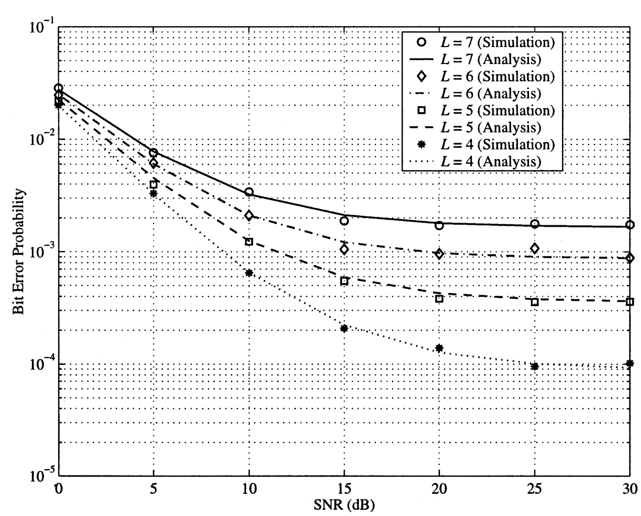


Fig. 5. BEP versus SNR for $N = 4$ reception branches, SIR $= 10$. The number of interferers varies from $L = 4$ to $L = 7$.

interference sources effectively. The interference suppression is not sensitive to the Gaussian assumption. In fact, it is well known that optimum combining maximizes the SINR, irrespective of the density function governing the interference.

## VI. Conclusions

In this paper, we derived a new closed-form expression of the exact BEP for optimum combining with BPSK modulation over a diversity channel with Rayleigh fading. The interference sources were assumed to have equal power and the Gaussian assumption was invoked for the aggregate of interference plus noise. However, it was shown that numerical results were insensitive to the Gaussian assumption for the interference. The theoretical BEP was developed starting from the optimum combining decision statistic expressed as a quadratic form of Gaussian random variables. The key result of this paper is contained in (27). The complexity of the new expression is relatively low, as it contains only finite sums and products. The theoretical results in the paper are amply demonstrated by simulations.

## Appendix A

## Derivation of the Characteristic Function

In this appendix, we derive the expression (8) for the characteristic function $\Phi_D(j\omega)$ of the decision statistic $D$.

From the signal model in Section II, and the definition of the whitened observation vector $\mathbf{x}$ and the modified channel vector $\mathbf{g}$, the following covariance matrix relations can be easily obtained: $\mathbf{R}_{xx} = E[\mathbf{x}\mathbf{x}^H] = P_s\mathbf{I}_N + \Lambda$, $\mathbf{R}_{gg} = \mathbf{I}_N$, $\mathbf{R}_{xg} = \mathbf{R}_{gx} = \sqrt{P_s}\mathbf{I}_N$.

Define $d_m = \lambda_m^{-1} g_m^* x_m + \lambda_m^{-1} g_m x_m^*$, then from (5) we have $D = \sum_{m=1}^{N} d_m$. To use the results in [13, App. B] to derive the characteristic function of $D$, we identify the following quantities using the notation in the reference: $X_m = x_m, Y_m = g_m$. Then with $\mathbf{R}_{xx}, \mathbf{R}_{gg}, \mathbf{R}_{xg}, \mathbf{R}_{gx}$ defined above, and [13, App. B, (B-5) and (B-6)], together with $A_m = B_m = 0$ and $C_m = \lambda_m^{-1}$, and after some manipulations, we obtain the characteristic function of $d_m$ as

$$\phi_{d_m}(j,\omega) = \frac{\lambda_m}{\left[\omega + j(\sqrt{P_s+\lambda_m} - \sqrt{P_s})\right]\left[\omega - j(\sqrt{P_s+\lambda_m} - \sqrt{P_s})\right]}. \tag{38}$$

Substituting (38) into $\Phi_D(j\omega) = \prod_{m=1}^{N} \phi_{d_m}(j\omega)$ and using $\lambda_{N_{\min}+1} = \lambda_{N_{\min}+2} = \ldots = \lambda_N = \sigma^2$, we obtain $\Phi_D(j\omega)$ as shown in (8).

## Appendix B

## Evaluation of $\sum_{m=1}^{N_{\min}} \int \ldots \int f_m(\mathbf{y}) p_{\mathbf{y}}(\mathbf{y}) d\mathbf{y}$

In this appendix, we evaluate the sum of integrals $\sum_{m=1}^{N_{\min}} \int \ldots \int f_m(\mathbf{y}) p_{\mathbf{y}}(\mathbf{y}) d\mathbf{y}$ and prove the relation in (33). The main steps of the evaluation are as follows.

1) Combine the sum of $N_{\min}$ integrals (each is an $N_{\min}$-fold integral) into one $N_{\min}$-fold integral.
2) Change integration limits to simplify evaluation of the integral.
3) For $N_{\min} > 1$, separate the $N_{\min}$-fold integration.

Due to page restrictions, we will present the procedure of the derivation but have to omit some details.

### A. Combine the Sum of $N_{\min}$ Integrals

Here our goal is to convert the sum of $N_{\min}$ integrals into one integral. We first consider the integrals $\int \ldots \int f_m(\mathbf{y}) p_{\mathbf{y}}(\mathbf{y}) d\mathbf{y}$ for $1 < m < N_{\min}$, then for $m = 1$ and $m = N_{\min}$.

For each $m$, $1 < m < N_{\min}$, we first carry out the integration over $y_m$, then over $y_{N_{\min}}$, next $y_{N_{\min}-1}$, and last over $y_1$. Recall that the values of the $y_m$'s, defined in (17), descend with the increase in index $m$. The integration limits of $y_m$ are $y_{m+1} \le y_m \le y_{m-1}$. Since $\eta \le y_{N_{\min}} \le \ldots \le y_2 \le y_1 < \infty$, $\int \ldots \int f_m(\mathbf{y}) p_{\mathbf{y}}(\mathbf{y}) d\mathbf{y}$ can be expressed as shown in (39) at the bottom of the page, with the inner integrations being carried out before the outer integrations.

Substituting $f_m(\mathbf{y})$ from (22) and $p_{\mathbf{y}}(\mathbf{y})$ from (25) into (39), and changing the variables from $y$ to $t$ as follows:

$$y_1 \to t_1, y_2 \to t_2, \ldots, y_{m-1} \to t_{m-1},$$
$$y_{m+1} \to t_m, \ldots, y_{N_{\min}} \to t_{N_{\min}-1}, y_m \to t_{N_{\min}}$$

we obtain an expression (which is omitted here to save space) of $\int \ldots \int f_m(\mathbf{y}) p_{\mathbf{y}}(y) dy$ for each $m$, $1 < m < N_{\min}$. Expressions for $m = 1$ and $m = N_{\min}$ can be obtained similarly. It can be shown that the sum of these expressions is

$$\begin{aligned}\sum_{m=1}^{N_{\min}} \int \cdots \int f_m(\mathbf{y}) p_{\mathbf{y}}(\mathbf{y}) d\mathbf{y} &= \frac{K_1}{2} z_0^{N-N_{\min}} \int_{\eta}^{\infty} \int_{\eta}^{t_1} \cdots \int_{\eta}^{t_{N_{\min}-2}} \\ &\quad \times \left\{ \prod_{n=1}^{N_{\min}-1} \left(1 - t_n^2\right) \left(t_n^2 - \eta^2\right)^{N_{\max}-N_{\min}} t_n \right. \\ &\quad \left. \times \exp\left[-\beta\left(t_n^2 - \eta^2\right)\right] \right\} \\ &\quad \times \left[ \prod_{1 \le i < j \le N_{\min}-1} \left(t_i^2 - t_j^2\right)^2 \right]\end{aligned}$$

$$\begin{aligned}&\int \cdots \int f_m(\mathbf{y}) p_{\mathbf{y}}(\mathbf{y}) d\mathbf{y} \\ &= \int_{\eta}^{\infty} \left\{ \int_{\eta}^{y_1} \cdots \left\{ \int_{\eta}^{y_{m-2}} \left\{ \int_{\eta}^{y_{m-1}} \cdots \left\{ \int_{\eta}^{y_{N_{\min}-1}} \left\{ \int_{y_{m+1}}^{y_{m-1}} f_m(\mathbf{y}) p_{\mathbf{y}}(\mathbf{y}) dy_m \right\} dy_{N_{\min}} \right\} \ldots dy_{m+1} \right\} dy_{m-1} \right\} \ldots dy_2 \right\} dy_1\end{aligned} \tag{39}$$

$$
\begin{aligned}
&\times \left\{ \int_{\eta}^{\infty} \left(1 - t_{N_{\min}}\right) \left(t_{N_{\min}}^2 - \eta^2\right)^{N_{\max}-N} \right.\\
&\qquad \times \left[ \prod_{n=1}^{N_{\min}-1} \left(t_{N_{\min}}^2 - t_n^2\right) \right]\\
&\qquad \left. \times \exp\left[-\beta \left(t_{N_{\min}}^2 - \eta^2\right)\right] dt_{N_{\min}} \right\}\\
&\times dt_{N_{\min}-1}, \ldots, dt_2 dt_1
\end{aligned} \tag{40}
$$

where $z_0 = 1 - \eta^2 = -1/\gamma$. Note that (40) consists of only one $N_{\min}$-fold integral.

To simplify notation, we perform the change of variables: $(t_n^2 - \eta^2) \rightarrow z_n$ for $n = 1, 2, \ldots, N_{\min} - 1$, then (40) becomes

$$
\begin{aligned}
&\sum_{m=1}^{N_{\min}} \int \cdots \int f_m(\mathbf{y}) p_{\mathbf{y}}(\mathbf{y}) d\mathbf{y}\\
&= \frac{K_1}{2^{N_{\min}}} z_0^{N-N_{\min}} \int_0^{\infty} \int_0^{z_1} \cdots \int_0^{z_{N_{\min}-2}}\\
&\quad \times \left\{ \prod_{n=1}^{N_{\min}-1} (z_0 - z_n) z_n^{N_{\max}-N_{\min}} \exp(-\beta z_n) \right\}\\
&\quad \times \left[ \prod_{1 \le i < j \le N_{\min}-1} (z_i - z_j)^2 \right]\\
&\quad \times \left\{ \int_{\eta}^{\infty} \left(1 - t_{N_{\min}}\right) \left(t_{N_{\min}}^2 - \eta^2\right)^{N_{\max}-N} \right.\\
&\qquad \times \left[ \prod_{n=1}^{N_{\min}-1} \left(t_{N_{\min}}^2 - \eta^2 - z_n\right) \right]\\
&\qquad \left. \times \exp\left[-\beta \left(t_{N_{\min}}^2 - \eta^2\right)\right] dt_{N_{\min}} \right\}\\
&\quad \times dz_{N_{\min}-1}, \ldots, dz_2 dz_1.
\end{aligned} \tag{41}
$$

### *B. Change Integration Limits*

Consider the integrations in (41). The integration limits for variables $z_i (1 \le i \le N_{\min} - 1)$ are (listed in the order of integration, starting with the innermost integral): $0 \le z_{N_{\min}-1} \le z_{N_{\min}-2}, \ldots, 0 \le z_3 \le z_2, 0 \le z_2 \le z_1, 0 \le z_1 < \infty$. We make the following observations.

1) The integrand is symmetric with respect to the variables $z_i$, $1 \le i \le N_{\min} - 1$ (i.e., for any $1 \le i, j \le N_{\min} - 1$, $i \neq j$, $z_i$, and $z_j$ can be interchanged, leaving the integrand the same).
2) There are $(N_{\min} - 1)!$ possible permutations of the integration limits of $z_i$ for which (41) will yield the same result. For example, one such permutation is $0 \le z_{N_{\min}-1} \le z_{N_{\min}-2}, \ldots, 0 \le z_3 \le z_1, 0 \le z_1 \le z_2$, $0 \le z_2 < \infty$. In this example, the order of integration is reversed for $z_1$ and $z_2$.
3) These $(N_{\min}-1)!$ integration limits are disjoint, and their union is the region $0 \le z_1 < \infty, 0 \le z_2 < \infty, 0 \le z_3 < \infty, \ldots, 0 \le z_{N_{\min}-1} < \infty$.

It follows that (41) is equal to the integrand integrated over $0 \le z_1, z_2, z_3, \ldots, z_{N_{\min}-1} < \infty$ and divided by $(N_{\min} - 1)!$

$$
\begin{aligned}
&\sum_{m=1}^{N_{\min}} \int \cdots \int fm(\mathbf{y}) p_{\mathbf{y}}(\mathbf{y}) d\mathbf{y}\\
&= \frac{1}{(N_{\min}-1)!} \frac{K_1}{2^{N_{\min}}} z_0^{N-N_{\min}} \int_0^{\infty} \int_0^{\infty} \cdots \int_0^{\infty}\\
&\quad \times \left\{ \prod_{n=1}^{N_{\min}-1} (z_0 - z_n) z_n^{N_{\max}-N_{\min}} \exp(-\beta z_n) \right\}\\
&\quad \times \left[ \prod_{1 \le i < j \le N_{\min}-1} (z_i - z_j)^2 \right]\\
&\quad \times \left\{ \int_{\eta}^{\infty} \left(1 - t_{N_{\min}}\right) \left(t_{N_{\min}}^2 - \eta\right)^{N_{\max}-N} \right.\\
&\qquad \times \left[ \prod_{n=1}^{N_{\min}-1} \left(t_{N_{\min}}^2 - \eta^2 - z_n\right) \right]\\
&\qquad \left. \times \exp\left[\beta \left(t_{N_{\min}}^2 - \eta^2\right)\right] dt_{N_{\min}} \right\}\\
&\quad \times dz_{N_{\min}-1}, \ldots, dz_2 dz_1.
\end{aligned} \tag{42}
$$

The integration above is treated differently for $N_{\min} = 1$ and $N_{\min} > 1$.

For $N_{\min} = 1$, (42) is a single integration. Using (26) and $z_0 = -1/\gamma$, we obtain for $N_{\min} = 1$

$$
\begin{aligned}
&\sum_{m=1}^{N_{\min}} \int \cdots \int f_m(\mathbf{y}) p_{\mathbf{y}}(\mathbf{y}) d\mathbf{y}\\
&\quad = \frac{1}{(N_{\max} - 1)!} \beta^{N_{\max}} \left(-\frac{1}{\gamma}\right)^{N-1}\\
&\qquad \times \left(B_{N_{\max}-N} - \frac{1}{2} \frac{(N_{\max} - N)!}{\beta^{N_{\max}-N+1}}\right)
\end{aligned} \tag{43}
$$

where

$$
B_q \triangleq \int_{\eta}^{\infty} \left(t_1^2 - \eta^2\right)^q \exp\left[-\beta \left(t_1^2 - \eta^2\right)\right] dt_1 \tag{44}
$$

for $q = 0, 1, 2, \ldots$. (28) can be easily derived from (44).

Next, we consider the case for $N_{\min} > 1$.

### *C. Separate the $N_{\min}$-Fold Integration for $N_{\min} > 1$*

It can be shown that an integration of an integrand which involves $[\prod_{1 \le i \le j \le N_{\min}-1} (z_i - z_j)^2]$ can be converted to an integration of integrand containing the determinant of a matrix [16, Ch. 3]. Exploring this result, from (42), we have

$$
\begin{aligned}
&\sum_{m=1}^{N_{\min}} \int \cdots \int f_m(\mathbf{y}) p_{\mathbf{y}}(\mathbf{y}) d\mathbf{y}\\
&\quad = \frac{K_1}{2^{N_{\min}}} z_0^{N-N_{\min}} \int_0^{\infty} \int_0^{\infty} \cdots \int_0^{\infty}\\
&\qquad \times \left\{ \prod_{n=1}^{N_{\min}-1} (z_0 - z_n) \exp(-\beta z_n) z_n^{N_{\max}-N_{\min}} \right\} \det \mathbf{Z}
\end{aligned}
$$

$$
\begin{aligned}
&\times \left\{ \int_{\eta}^{\infty} \left(1 - t_{N_{\min}}\right) \left(t_{N_{\min}}^2 - \eta^2\right)^{N_{\max} - N} \right. \\
&\qquad \times \left[ \prod_{n=1}^{N_{\min}-1} \left(t_{N_{\min}}^2 - \eta^2 - z_n\right) \right] \\
&\qquad \left. \times \exp\left[-\beta \left(t_{N_{\min}}^2 - \eta^2\right)\right] dt_{N_{\min}} \right\} \\
&\times dz_{N_{\min}-1}, \ldots, dz_2 dz_1
\end{aligned} \tag{45}
$$

where $\det \mathbf{Z}$ is the determinant of the matrix $\mathbf{Z}$, whose $i$th row, $j$th column element is $z_j^{i+j-2}$. Note that all the elements on the $j$th column of the matrix $\mathbf{Z}$ depend only on variable $z_j$.

Obviously, we can express $\prod_{n=1}^{2}(z_0 - z_n)$ as

$$
\prod_{n=1}^{2}(z_0 - z_n) = \sum_{p=0}^{2}(-1)^{2-p} z_0^p \sum_{\substack{m_1+m_2=2-p \\ m_1, m_2 \in \{0,1\}}} z_1^{m_1} z_2^{m_2}. \tag{46}
$$

The second summation is taken over all sets of $m_1$ and $m_2$ satisfying the stated conditions. Expanding the two products in (45) similarly, we get

$$
\begin{aligned}
&\sum_{m=1}^{N_{\min}} \int \cdots \int f_m(\mathbf{y}) p_{\mathbf{y}}(\mathbf{y}) d\mathbf{y} \\
&= \frac{K_1}{2^{N_{\min}}} z_0^{N - N_{\min}} \sum_{p=0}^{N_{\min}-1} (-1)^{N_{\min}-1-p} z_0^p \\
&\quad \times \sum_{q=0}^{N_{\min}-1} (-1)^{N_{\min}-1-q} \sum_{\substack{m_1+\cdots+m_{N_{\min}-1}=N_{\min}-1-p \\ m_i \in \{0,1\}}} \\
&\quad \times \sum_{\substack{n_1+\cdots+n_{N_{\min}-1}=N_{\min}-1-q \\ n_i \in \{0,1\}}} \int_0^{\infty} \int_0^{\infty} \cdots \int_0^{\infty} \\
&\quad \times \left\{ \prod_{j=1}^{N_{\min}-1} z_j^{N_{\max}-N_{\min}+m_j+n_j} \exp(-\beta z_j) \right\} \\
&\quad \times \det \mathbf{Z} dz_{N_{\min}-1}, \ldots, dz_2 dz_1 \\
&\quad \times \left\{ \int_{\eta}^{\infty} \left(1 - t_{N_{\min}}\right) \left(t_{N_{\min}}^2 - \eta^2\right)^{N_{\max}-N+q} \right. \\
&\qquad \left. \times \exp\left[-\beta \left(t_{N_{\min}}^2 - \eta^2\right)\right] dt_{N_{\min}} \right\}.
\end{aligned} \tag{47}
$$

Since the elements on the $j$th column of matrix $\mathbf{Z}$ depend only on variable $z_j$, we can multiply $z_j^{N_{\max}-N_{\min}+m_j+n_j} \exp(-\beta z_j)$ (for $j = 1, 2, \ldots, N_{\min}-1$) with all elements of the $j$th column of $\mathbf{Z}$, then integrate the $j$th column with respect to $z_j$ before calculating the determinant. In this way, we separate the $N_{\min}$-fold integration into independent integrations. Carrying out these straightforward integrations and substituting $K_1$ from (26) and $z_0 = -1/\gamma$ in (47), after some manipulations, we obtain (33).

When $N_{\min} = 1$, if we assume $\det \mathbf{W} = 1$, then it can be shown that (33) yields the same result as (43). Therefore, we can use (33) for $N_{\min} = 1$ as well as for $N_{\min} > 1$.

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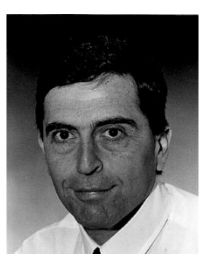


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