New Exact Closed-Form Expression of Bit Error Probability for Optimum Combining

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Abstract — **A new closed-form expression is derived for the *exact* bit error probability (BEP) for optimum combining with binary phase shift keying. The exact BEP expression is for multiple, equal power, co-channel interferers and multiple reception branches, with the number of interferers less than the number of reception branches. The aggregate interference and noise is modeled as Gaussian distributed. Both the desired signal and interference are subject to Rayleigh fading. The derivation starts by expressing the optimum combining decision statistic as a sum of quadratic forms of Gaussian random variables and it proceeds to average over the fading interference. The new BEP expression has low complexity as it contains only finite sums and products.**

I. Introduction

Optimum combining is a well-known method to combat fading and suppress co-channel interference in wireless communication systems with reception diversity. It combines the outputs of the reception branches in an optimum way and achieves the maximum output signal-to-interference-plus-noise ratio (SINR).

Performance analysis of optimum combining has been an active research area. Analysis for the case of a single interference source can be found in [1] and [2]. In [1], Rayleigh fading is assumed for the desired signal, but mean values, rather than actual distributions, are used to represent fading effects on the interference. In [2], exact expressions (requiring integration) are developed under the assumption of Rayleigh fading for both the desired signal and interference. Closed-form expressions of the bit error probability (BEP) for this case were obtained in [3]. By 'closed-form' we mean expressions that contain only analytical functions and no integrals.

The case of multiple interferers is more challenging. Closed-form expressions of the BEP for the number of interferers no less than the number of reception branches and negligible thermal noise with binary phase shift keying (BPSK) modulation were developed in [4]. To our knowledge, no exact, closed-form expressions have been published for Rayleigh fading diversity channels, BPSK or M-ary phase shift keying (MPSK) with multiple interferers and non-negligible thermal noise. The performance of such systems has been studied extensively through the use of Monte Carlo simulations [1], capacity [5], upper bound [6], approximate expressions [7], and exact expressions that contain integrals [8]. A summary of recent results on optimum combining can be found in [9].

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The conventional way of deriving the expression for symbol error probability (SEP) or BEP starts with deriving the probability density function of the SINR conditioned on channel realizations of the desired signal and the interference. For an exact conditional SEP, the Gaussian assumption is necessary for the interference plus noise. The unconditional SEP is obtained by averaging first over the desired signal channel and then averaging over the interference channel. Since it involves two steps of averaging, a closed-form expression for SEP is difficult to obtain.

In this paper, we take a different approach by performing the analysis directly on the decision statistic rather than on the SINR. We show that for BPSK, this approach allows exact BEP analysis and it requires only averaging over the fading of the interference. Although the algebra is somewhat cumbersome, at the end, this method provides a closed-form expression. In this work, we assume that the number of interferers is less than the number of reception branches. The case of more interferers than reception branches is left for a future publication.

The paper is organized as follows. Followed the system model in Section II, we derive the conditional BEP in Section III. In Section IV, we average the conditional BEP over the fading of the interference to get the unconditional BEP. Numerical results are presented in section V and finally, conclusions are drawn in Section VI.

II. System Model

Consider a wireless communications system with N independent reception branches and $L+1$ ($L<N$) users. Of the users, user 1 is the desired user and it transmits signals with power P_s. The other L sources are considered interference. Assuming perfect carrier demodulation and synchronization, the sampled output of the matched filter for the l-th branch at time k is

$$r_{k,l} = \sqrt{P_s} c_l s_k + \sum_{i=1}^{L} \sqrt{P_I} c_{i,l} s_{i,k} + n_{k,l},\ l = 1, 2, \cdots, N, \quad (1)$$

where c_l and s_k are respectively, the channel gain and BPSK symbol of the desired signal; $c_{i,l}$ and $s_{i,k}$ are respectively, the i-th interferer's channel gain and symbol; P_I is the interference power (assumed equal for all interference sources). The term $n_{k,l}$ represents additive white Gaussian noise (AWGN). The channel gains c_l and $c_{i,l}$ are assumed to be independently and identically distributed (i.i.d.), zero-mean, circularly symmetric, complex Gaussian random variables (Rayleigh fading), with variance 1/2 per dimension. The signal model in vector notation is

$$\mathbf{r}_k = \sqrt{P_s}\mathbf{c}s_k + \sqrt{P_I}\sum_{i=1}^{L}\mathbf{c}_i s_{i,k} + \mathbf{n}_k, \tag{2}$$

where $\mathbf{r}_k = [r_{k,1}, r_{k,2}, \cdots, r_{k,N}]^T$, $\mathbf{c} = [c_1, c_2, \cdots, c_N]^T$, $\mathbf{c}_i = [c_{i,1}, c_{i,2}, \cdots, c_{i,N}]^T$, $\mathbf{n}_k = [n_{k,1}, n_{k,2}, \cdots, n_{k,N}]^T$, and the superscript T denotes vector transposition.

Define the interference plus noise vector as $\mathbf{z}_k = \sqrt{P_I}\sum_{i=1}^{L}\mathbf{c}_i s_{i,k} + \mathbf{n}_k$, then (2) becomes

$$\mathbf{r}_k = \sqrt{P_s}\mathbf{c}s_k + \mathbf{z}_k. \tag{3}$$

It is further assumed that conditioned on the vectors $\mathbf{c}_i$, the interference plus noise vector $\mathbf{z}_k$ has a multivariate complex-Gaussian distribution with zero mean and covariance matrix $\mathbf{R} = E\left[\mathbf{z}_k\mathbf{z}_k^H\right]$,

$$\mathbf{R} = P_I\sum_{i=1}^{L}\mathbf{c}_i\mathbf{c}_i^H + \sigma^2\mathbf{I}_N, \tag{4}$$

where the superscript H denotes the Hermitian transposition, σ^2 is the power of the noise, and $\mathbf{I}_N$ is an identity matrix of rank N.

Assuming that the vectors $\mathbf{c}_i$ (for $i = 1, 2, \cdots, L$) are linearly independent (a reasonable assumption since the components of these vectors are realizations of mutually independent random variables), the N dimensional matrix $P_I\sum_{i=1}^{L}\mathbf{c}_i\mathbf{c}_i^H$ has rank L. That means that it has $(N - L)$ zero eigenvalues. Adopting a convention whereby eigenvalues are listed in descending order, the zero eigenvalues of $P_I\sum_{i=1}^{L}\mathbf{c}_i\mathbf{c}_i^H$ are indexed $m = L+1, L+2, \cdots, N$. Let the corresponding eigenvectors be denoted as $\mathbf{u}_m$. From the eigenvalue relation $P_I\sum_{i=1}^{L}\mathbf{c}_i\mathbf{c}_i^H\mathbf{u}_m = 0\cdot\mathbf{u}_m$, we clearly have $\sum_{i=1}^{L}\mathbf{c}_i\mathbf{c}_i^H\mathbf{u}_m = 0$ for $m = L+1, \cdots, N$. It follows that the vectors $\mathbf{u}_m$ can also serve as eigenvectors of the matrix $\mathbf{R}$. Indeed we have

$$\begin{aligned}\mathbf{R}\mathbf{u}_m &= P_I\sum_{i=1}^{L}\mathbf{c}_i\mathbf{c}_i^H\mathbf{u}_m + \sigma^2\mathbf{u}_m \\ &= \sigma^2\mathbf{u}_m, \quad m = L+1, \cdots, N.\end{aligned} \tag{5}$$

Therefore the interference plus noise covariance matrix $\mathbf{R}$ has $(N - L)$ eigenvalues equal to σ^2. We also conclude that only the L largest eigenvalues of $\mathbf{R}$ are dependent on the channel vectors $\mathbf{c}_i$, $i = 1, 2, \cdots, L$.

Diagonalize $\mathbf{R}$ as $\mathbf{R} = \mathbf{U}\Lambda\mathbf{U}^H$, where Λ = diag $(\lambda_1, \lambda_2, \cdots, \lambda_N)$, λ_1,λ_2, $\cdots$,λ_N are the eigenvalues of $\mathbf{R}$, and $\mathbf{U}$ is the unitary matrix whose columns are the eigenvectors of $\mathbf{R}$. Since the channel vectors $\mathbf{c}_i$ are modeled as random, it follows that in our analysis, $\lambda_1 \geqslant \lambda_2 \geqslant \cdots \geqslant \lambda_L$ are random variables, while $\lambda_m = \sigma^2$ for $m = L+1, L+2, \cdots, N$. For later use, we denote the set of non-trivial eigenvalues $\lambda = \{\lambda_1, \lambda_2, \cdots, \lambda_L\}$. The inverse covariance matrix of $\mathbf{R}$ is $\mathbf{R}^{-1} = \mathbf{U}\Lambda_z^{-1}\mathbf{U}^H$.

III. Derivation of Conditional BEP

In this and the next section, we carry out the theoretical analysis of the BEP of optimum combining for BPSK modulation in the presence of multiple interference sources when both the desired signal and interference are subject to Rayleigh fading and the number of interference sources is less than that of diversity branches.

With the optimum combining detector, the received signal vector $\mathbf{r}_k$ is weighted and combined to obtain the output signal. The weight vector that yields the maximum SINR is [1]

$$\mathbf{w} = \mathbf{R}^{-1}\mathbf{c}. \tag{6}$$

The output of the combiner is $\mathbf{w}^H\mathbf{r}_k$. For BPSK modulation, the decision rule of the detector is: if $\mathrm{Re}(\mathbf{w}^H\mathbf{r}_k) \geqslant 0$, the decision is made that 1 is transmitted; otherwise the decision is made that -1 is transmitted. Due to the symmetry of the BPSK constellation and assuming a source with equal symbol probabilities, it suffices to analyze the case of $s_k = 1$. Then from (3), the received signal is

$$\mathbf{r}_k = \sqrt{P_s}\mathbf{c} + \mathbf{z}_k. \tag{7}$$

Define

$$D = 2\mathrm{Re}\left(\mathbf{w}^H\mathbf{r}_k\right) = \mathbf{w}^H\mathbf{r}_k + \left(\mathbf{w}^H\mathbf{r}_k\right)^*. \tag{8}$$

According to the decision rule, when $D < 0$, the decision is made that -1 is transmitted and an error occurs. Therefore the BEP is $P_e = \Pr(D < 0)$. The analysis has two steps. First, the BEP is expressed conditioned on the fading of the interference. Subsequently, the conditioned BEP is averaged over the fading of the interference.

Fixing the values of the channels $\mathbf{c}_i$ of the interference sources, leads to fixed values of eigenvalues of interference plus noise covariance matrix $\mathbf{R}$. These eigenvalues λ_m form the diagonal of the matrix Λ. Substituting (6) into (8) and de-composing $\mathbf{R}^{-1}$ as $\mathbf{U}\Lambda_z^{-1}\mathbf{U}^H$, we can express D as

$$D = \sum_{m=1}^{N}\lambda_m^{-1}\left(g_m^* x_m + g_m x_m^*\right), \tag{9}$$

where λ_m's are the eigenvalues of $\mathbf{R}$ defined previously, x_m's are elements of the whitened observation vector $\mathbf{x} = [x_1, x_2, \cdots, x_N]^T = \mathbf{U}^H\mathbf{r}_k$, and g_m's are elements of the whitened channel vector $\mathbf{g} = [g_1, g_2, \cdots, g_N]^T = \mathbf{U}^H\mathbf{c}$. Since $\mathbf{x} = \sqrt{P_s}\mathbf{g} + \mathbf{U}^H\mathbf{z}_k$, conditioned on the eigenvalues λ_m, the variable D is a quadratic form of Gaussian random variables. Our goal is to evaluate the conditional BEP $P(e|\lambda) = \Pr(D < 0|\lambda)$, where the notation indicates the dependency on the L largest eigenvalues of $\mathbf{R}$ (the other $(N - L)$ eigenvalues are equal to constant σ^2).

Let $\Phi_D(j\omega)$ be the characteristic function of D conditioned on λ. Using results from [10, Appendix B], it can be shown that the conditional BEP is

$$\begin{aligned}P(e|\lambda) &= -\frac{1}{2\pi j}\int_{-\infty+j\varepsilon}^{\infty+j\varepsilon}\frac{\Phi_D(j\omega)}{\omega}d\omega \\ &= -\sum_{\mathrm{Im}(\omega_m)>0}\mathrm{Res}\left[\frac{\Phi_D(j\omega)}{\omega};\omega_m\right],\end{aligned} \tag{10}$$

where ε is a small positive number and $\mathrm{Res}\left[\frac{\Phi_D(j\omega)}{\omega};\omega_m\right]$ denotes the residue of $\Phi_D(j\omega)/\omega$ at pole ω_m. The summation is taken over the poles in the upper half of the complex plane.

In Appendix A it is shown that the characteristic function

$\Phi_D(j\omega)$ can be expressed as

$$\begin{aligned}\Phi_D(j\omega) &= \left\{\frac{\sigma^2}{\left[\omega + j\left(\sqrt{P_s+\sigma^2}-\sqrt{P_s}\right)\right]}\right.\\ &\left.\frac{1}{\left[\omega - j\left(\sqrt{P_s+\sigma^2}+\sqrt{P_s}\right)\right]}\right\}^{N-L}\\ &\prod_{m=1}^{L}\left\{\frac{\lambda_m}{\left[\omega + j\left(\sqrt{P_s+\lambda_m}-\sqrt{P_s}\right)\right]}\right.\\ &\left.\frac{1}{\left[\omega - j\left(\sqrt{P_s+\lambda_m}+\sqrt{P_s}\right)\right]}\right\}. \quad (11)\end{aligned}$$

Substituting (11) into (10), and carrying out the calculation of the residues, we obtain the conditional BEP as (12), which is shown on the top of next page.

IV. Derivation of Unconditional BEP

For the general case with interference, the unconditional BEP P_e is obtained by averaging the conditional BEP $P(e|\lambda)$ over the fading of the interference $\mathbf{c}_i$, or equivalently over the eigenvalues $\lambda = \{\lambda_1, \lambda_2, \cdots, \lambda_L\}$,

$$P_e = \int P(e|\lambda) p_\lambda(\lambda) d\lambda, \quad (13)$$

where we abused the notation a bit to indicate a multiple-fold integral, and $p_\lambda(\lambda)$ is the joint probability density function of the eigenvalues $\lambda = \{\lambda_1, \lambda_2, \cdots, \lambda_L\}$. The density $p_\lambda(\lambda)$ was developed in [8] for a signal model similar to ours and is given by

$$\begin{aligned}p_\lambda(\lambda) &= K_0 \frac{1}{P_I^L}\left[\prod_{i=1}^{L}\exp\left(-\frac{\lambda_i-\sigma^2}{P_I}\right)\left(\frac{\lambda_i-\sigma^2}{P_I}\right)^{N-L}\right]\\ &\times\left[\prod_{1\le i<j\le L-1}\left(\frac{\lambda_i-\sigma^2}{P_I}-\frac{\lambda_j-\sigma^2}{P_I}\right)^2\right], \quad (14)\end{aligned}$$

where

$$K_0 = \frac{1}{\prod_{i=1}^{L}(N-i)!\prod_{i=1}^{L}(L-i)!}. \quad (15)$$

The conditional BEP in (12) is a non-rational function of the eigenvalues λ_m's. To facilitate the integration in (13), we define the following transformation of variables

$$y_m = \sqrt{\frac{\lambda_m}{P_s}+1}, \qquad m = 1, 2, \cdots, L \quad (16)$$

and define the set $\mathbf{y} = \{y_1, y_2, \cdots, y_L\}$. Since λ_m is random, y_m is random as well. Also define

$$\eta = \sqrt{\frac{\sigma^2}{P_s}+1} = \sqrt{\frac{1+\gamma}{\gamma}}. \quad (17)$$

From (16) and (17), we have

$$\begin{aligned}\lambda_m &= P_s\left(y_m^2-1\right) \quad m = 1, 2, \cdots, L \quad (18)\\ \sigma^2 &= P_s\left(\eta^2-1\right). \quad (19)\end{aligned}$$

Substituting (18) and (19) into (12), and after some straightforward manipulations, we get the conditional BEP as a function of the variables y_m's,

$$\begin{aligned}&P(e|\mathbf{y})\\ &= -\sum_{m=1}^{L} f_m(\mathbf{y}) + (-1)^{N-L}\sum_{l=0}^{N-L-1}\binom{N-L+l-1}{l}\\ &\left[1+\sum_{m=1}^{L} h_{m,l}(\mathbf{y})\right]\frac{(1-\eta)^{N-L-l}}{(2\eta)^{N-L+l}}\left(-\frac{1}{\gamma}\right)^l, \quad (20)\end{aligned}$$

where the functions $f_m(\mathbf{y})$ and $h_{m,l}(\mathbf{y})$ are defined respectively,

$$\begin{aligned}f_m(\mathbf{y}) &= \frac{1-y_m}{2y_m}\frac{\left(1-\eta^2\right)^{N-L}}{\left(y_m^2-\eta^2\right)^{N-L}}\left\{\prod_{n=1}^{m-1}\frac{1-y_n^2}{y_m^2-y_n^2}\right\}\\ &\left\{\prod_{n=m+1}^{L}\frac{1-y_n^2}{y_m^2-y_n^2}\right\} \quad (21)\end{aligned}$$

and

$$\begin{aligned}h_{m,l}(\mathbf{y}) &= (-1)^{N-L-l}\frac{(1+\eta)^{N-L-l}}{2y_m}\frac{1}{\left(y_m^2-\eta^2\right)^{N-L-l}}\\ &b_{N-L-l}(y_m)\left\{\prod_{n=1}^{m-1}\frac{1-y_n^2}{y_m^2-y_n^2}\right\}\\ &\left\{\prod_{n=m+1}^{L}\frac{1-y_n^2}{y_m^2-y_n^2}\right\}. \quad (22)\end{aligned}$$

The function $b_k(y_m)$ in (22) is in turn defined for $1 \le k \le N-L$,

$$b_k(y_m) = -(1+y_m)(\eta-y_m)^k + (1-y_m)(\eta+y_m)^k. \quad (23)$$

Clearly, the conditional BEP $P(e|\mathbf{y})$ is a rational function of the elements of the set $\mathbf{y}$. Using the Jacobian of the transformation from λ to $\mathbf{y}$, we get the joint probability density function of $\mathbf{y}$ as

$$\begin{aligned}p_{\mathbf{y}}(\mathbf{y}) &= p_\lambda(\lambda)\frac{d\lambda_1}{dy_1}\frac{d\lambda_2}{dy_2}\cdots\frac{d\lambda_L}{dy_L}\Big|_{\lambda=\mathbf{y}}\\ &= K_1\left\{\prod_{i=1}^{L}\exp\left[-\beta\left(y_i^2-\eta^2\right)\right]\left(y_i^2-\eta^2\right)^{N-L}\right\}\\ &\times\left[\prod_{1\le i<j\le L-1}\left(y_i^2-y_j^2\right)^2\right] y_1 y_2\cdots y_L \quad (24)\end{aligned}$$

for $y_1 \geqslant y_2 \geqslant \cdots \geqslant y_L \geqslant \eta$, where $\beta = P_s/P_I$ is the signal to interference ratio (SIR) and

$$K_1 = \frac{2^L}{\left[\prod_{i=1}^{L}(N-i)!\right]\left[\prod_{i=1}^{L}(L-i)!\right]}\beta^{NL}. \quad (25)$$

The unconditional BEP P_e is obtained by averaging the

$$
\begin{aligned}
P(e|\lambda) \;=\;& (-1)^{N+1}\sum_{m=1}^{L}\frac{\lambda_m}{2\sqrt{P_s+\lambda_m}\left(\sqrt{P_s+\lambda_m}+\sqrt{P_s}\right)}\frac{\left(\sigma^2\right)^{N-L}}{\left(\lambda_m-\sigma^2\right)^{N-L}}\prod_{n=1}^{m-1}\frac{\lambda_n}{(\lambda_m-\lambda_n)}\prod_{n=m+1}^{L}\frac{\lambda_n}{(\lambda_m-\lambda_n)}\\
&+\left(\sigma^2\right)^{N-L}\sum_{l=0}^{N-L-1}\binom{N-L+l-1}{l}\left\{1+\frac{1}{2}(-1)^L\left(\sqrt{P_s+\sigma^2}+\sqrt{P_s}\right)^{N-L-l}\left(\prod_{n=1}^{L}\lambda_n\right)\right.\\
&\sum_{m=1}^{L}\left(\prod_{n=1}^{m-1}\frac{1}{\lambda_m-\lambda_n}\right)\left(\prod_{n=m+1}^{L}\frac{1}{\lambda_m-\lambda_n}\right)\frac{1}{\sqrt{P_s+\lambda_m}}\left[\frac{1}{\left(\sqrt{P_s+\lambda_m}-\sqrt{P_s}\right)}\right.\\
&\left.\left.\frac{1}{\left(\sqrt{P_s+\sigma^2}+\sqrt{P_s+\lambda_m}\right)^{N-L-l}}+\frac{1}{\left(\sqrt{P_s+\lambda_m}+\sqrt{P_s}\right)}\frac{1}{\left(\sqrt{P_s+\sigma^2}-\sqrt{P_s+\lambda_m}\right)^{N-L-l}}\right]\right\}\\
&\frac{1}{\left(\sqrt{P_s+\sigma^2}+\sqrt{P_s}\right)^{N-L-l}\left(2\sqrt{P_s+\sigma^2}\right)^{N-L+l}}. \qquad (12)
\end{aligned}
$$

conditional $P(e|\mathbf{y})$ over the random variables in the set $\mathbf{y}$,

$$
\begin{aligned}
P_e \;=\;& \int P(e|\mathbf{y})\,p_{\mathbf{y}}(\mathbf{y})d\mathbf{y}\\
=\;& -\sum_{m=1}^{L}\int f_m(\mathbf{y})\,p_{\mathbf{y}}(\mathbf{y})d\mathbf{y}+\\
& (-1)^{N-L}\sum_{l=0}^{N-L-1}\binom{N-L+l-1}{l}\left(-\frac{1}{\gamma}\right)^{l}\\
& \left[1+\sum_{m=1}^{L}\int h_{m,l}(\mathbf{y})\,p_{\mathbf{y}}(\mathbf{y})d\mathbf{y}\right]\\
& \frac{(1-\eta)^{N-L-l}}{(2\eta)^{N-L+l}}. \qquad (26)
\end{aligned}
$$

We proceed to evaluate the terms of (26).

A Computation of $\sum_{m=1}^{L}\int f_m(\mathbf{y})\,p_{\mathbf{y}}(\mathbf{y})d\mathbf{y}$

The following definitions are needed:

1. B_q is a sequence defined as

$$B_0 = \sqrt{\frac{\pi}{\beta}}\exp\left(\beta\eta^2\right)Q\left(\sqrt{2\beta}\eta\right) \qquad (27)$$

$$B_1 = \frac{\eta}{2\beta}+\left(\frac{1}{2\beta}-\eta^2\right)B_0; \qquad (28)$$

for $q \geq 2$,

$$B_q = \left(\frac{2q-1}{2\beta}-\eta^2\right)B_{q-1}+\frac{(q-1)}{\beta}\eta^2 B_{q-2} \qquad (29)$$

2. $H_{p,q}$ is a function of integers p and q. For $0 \leq p,q \leq L-1$,

$$
\begin{aligned}
H_{p,q} \;=\;& \frac{1}{\left[\prod_{i=1}^{L}(N-i)!\right]\left[\prod_{i=1}^{L}(L-i)!\right]}\\
& \sum_{\substack{m_1+\cdots+m_{L-1}=L-1-p\\ m_i\in\{0,1\}}}\;\sum_{\substack{n_1+\cdots+n_{L-1}=L-1-q\\ n_i\in\{0,1\}}}\\
& \det\mathbf{W}, \qquad (30)
\end{aligned}
$$

where for $L=1$, $\det\mathbf{W}=1$; for $L>1$, $\det\mathbf{W}$ is the the determinant of an $(L-1)\times(L-1)$ matrix whose i-th row, j-th column element is

$$W_{i,j} = (m_j+n_j+N-L+i+j-2)!. \qquad (31)$$

The summations in (30) are taken over all sets of indices satisfying the stated conditions.

Using these definitions, it can be shown that [11]:

$$
\begin{aligned}
& \sum_{m=1}^{L}\int f_m(\mathbf{y})\,p_{\mathbf{y}}(\mathbf{y})d\mathbf{y}\\
=\;& \left(-\frac{1}{\gamma}\right)^{N-L}\beta^{N-L+1}\sum_{p=0}^{L-1}\sum_{q=0}^{L-1}(-1)^q H_{p,q}\\
& \frac{1}{\gamma^p}\left(B_q-\frac{1}{2}\frac{q!}{\beta^{q+1}}\right)\beta^{p+q}. \qquad (32)
\end{aligned}
$$

B Computation of $\sum_{m=1}^{L}\int h_{m,l}(\mathbf{y})\,p_{\mathbf{y}}(\mathbf{y})d\mathbf{y}$

By mathematical induction it can be proved that the function $b_k(y_m)$ defined in (23) can be alternatively expressed as

$$b_k(y_m) = 2y_m\sum_{t=0}^{[k/2]}a_{k,t}\left(y_m^2-\eta^2\right)^t, \qquad (33)$$

where $[k/2]$ denotes the largest integer that is equal to or less than $k/2$, and $a_{k,t}$ is evaluated as:

$$a_{k,t} = \left[\binom{k-1-t}{t}(1-\eta)-2\eta\binom{k-1-t}{t-1}\right](2\eta)^{k-1-2t}. \qquad (34)$$

When calculating $a_{k,t}$, we assume $\binom{m}{n}=0$ for $m<n$ or $n<0$.

Using steps similar to those in [11], it can be shown that:

$$\begin{aligned}
&\sum_{m=1}^{L} \int h_{m,l}(\mathbf{y})\, p_{\mathbf{y}}(\mathbf{y}) d\mathbf{y} \\
&= (-1)^{N-L-l} (1+\eta)^{N-L-l} \beta^{N-L+1} \\
&\quad \sum_{t=0}^{[(N-L-l)/2]} a_{N-L-l,t} \frac{1}{\beta^{l+t+1}} \\
&\quad \sum_{p=0}^{L-1} \sum_{q=0}^{L-1} (-1)^q H_{p,q} (l+t+q)! \left(\frac{\beta}{\gamma}\right)^p . \qquad (35)
\end{aligned}$$

Using the expressions obtained in (32) and (35), we can proceed to evaluate (26) to obtain the exact BEP for any given number of diversity branches N, number of interferers L ($L < N$), SNR $\gamma = P_s/\sigma^2$ and SIR $\beta = P_s/P_I$.

V. Numerical Results

As mentioned in the introduction, the performance of optimum combining has been evaluated in several publications (without the benefit of exact closed-form expressions). Our focus in this paper is not the performance of optimum combining, rather we use Monte Carlo simulations to demonstrate the new exact BEP expression.

Fig. 1, 2 and 3 show the BEP versus SNR for different SIR β. Fig. 1 and 2 are for $N = 4$ diversity branches, $L = 2$ and 3 interferers, respectively. Fig. 3 is for $N = 8$ diversity branches and $L = 5$ interferers. In Fig. 2 and 3, the interference generated in the simulations had a Gaussian distribution as assumed in developing the BEP analysis. Simulation results in Fig. 1 were generated for two interference sources transmitting BPSK symbols. Analytical results were calculated using (26) and the related expressions such as (32) and (35).

In all the figures, the analysis results match the simulation results. This provides convincing demonstration of the validity of the analytical expression for BEP.

As shown by Fig. 1 for BPSK interference, the Gaussian assumption for the interference, while necessary for obtaining the theoretical results, is not critical for the accuracy of the BEP expressions. This can be explained by recognizing that the system has a sufficient number of degrees of freedom to suppress the interference sources effectively. The interference suppression is not sensitive to the Gaussian assumption. In fact, it is well known that optimum combining maximizes the SINR irrespective of the density function governing the interference.

VI. Conclusions

In this paper, we derived a new closed-form expression of the *exact* BEP for optimum combining with BPSK modulation over a diversity channel with Rayleigh fading. The number of diversity branches was assumed larger than the number of interference sources. The interference sources were assumed to have equal power and the Gaussian assumption was invoked for the aggregate of interference plus noise. However, it was shown that numerical results were insensitive to the Gaussian assumption for the interference. The complexity of the new expression is relatively low as it contains only finite sums and products. The theoretical results in the paper are amply demonstrated by simulations.

Appendix A: Derivation of the Characteristic Function

In this appendix, we derive the expression (11) for the characteristic function $\Phi_D(j\omega)$ of the test statistic D. Define

$$d_m = \lambda_m^{-1} g_m^* x_m + \lambda_m^{-1} g_m x_m^*, \qquad (36)$$

then from (9) we have

$$D = \sum_{m=1}^{N} d_m. \qquad (37)$$

From the signal model in Section II, and the definition of the whitened observation vector $\mathbf{x}$ and the modified channel vector $\mathbf{g}$, the following covariance matrix relations can be easily obtained:

$$\begin{aligned}
\mathbf{R}_{xx} &= E\left[\mathbf{x}\mathbf{x}^H\right] = P_s \mathbf{I}_N + \Lambda \qquad (38)\\
\mathbf{R}_{gg} &= \mathbf{I}_N \qquad (39)\\
\mathbf{R}_{xg} &= \mathbf{R}_{gx} = \sqrt{P_s}\mathbf{I}_N. \qquad (40)
\end{aligned}$$

To use the results in [10, Appendix B] to derive the characteristic function, we identify the following quantities using the notation in the reference: $X_m = x_m$, $Y_m = g_m$. Then using (38) to (40), and (B-6) and (B-5) in [10, Appendix B], together with $A_m = B_m = 0$ and $C_m = \lambda_m^{-1}$, and after some manipulations, we obtain the characteristic function of d_m as

$$\begin{aligned}
\phi_{d_m}(j\omega) &= \frac{\lambda_m}{\left[\omega + j\left(\sqrt{P_s + \lambda_m} - \sqrt{P_s}\right)\right]} \\
&\quad \frac{\lambda_m}{\left[\omega - j\left(\sqrt{P_s + \lambda_m} + \sqrt{P_s}\right)\right]}. \qquad (41)
\end{aligned}$$

It follows that the characteristic function of D is

$$\begin{aligned}
\Phi_D(j\omega) &= \prod_{m=1}^{N} \phi_{d_m}(j\omega) \\
&= \prod_{m=1}^{N} \frac{\lambda_m}{\left[\omega - j\left(\sqrt{P_s + \lambda_m} + \sqrt{P_s}\right)\right]} \\
&\quad \frac{1}{\left[\omega - j\left(\sqrt{P_s + \lambda_m} + \sqrt{P_s}\right)\right]}. \qquad (42)
\end{aligned}$$

Since $\lambda_{L+1} = \lambda_{L+2} = \cdots = \lambda_N = \sigma^2$,

$$\begin{aligned}
\Phi_D(j\omega) &= \left\{ \frac{\sigma^2}{\left[\omega + j\left(\sqrt{P_s + \sigma^2} - \sqrt{P_s}\right)\right]} \right. \\
&\quad \left. \frac{1}{\left[\omega - j\left(\sqrt{P_s + \sigma^2} + \sqrt{P_s}\right)\right]} \right\}^{N-L} \\
&\quad \prod_{m=1}^{L} \left\{ \frac{\lambda_m}{\left[\omega + j\left(\sqrt{P_s + \lambda_m} - \sqrt{P_s}\right)\right]} \right. \\
&\quad \left. \frac{1}{\left[\omega - j\left(\sqrt{P_s + \lambda_m} + \sqrt{P_s}\right)\right]} \right\}. \qquad (43)
\end{aligned}$$

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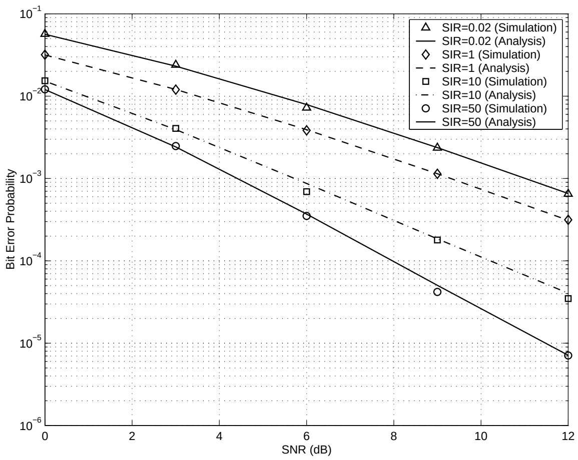

Fig. 1: BEP versus SNR for $N = 4$ branches, $L = 2$ BPSK interferers.

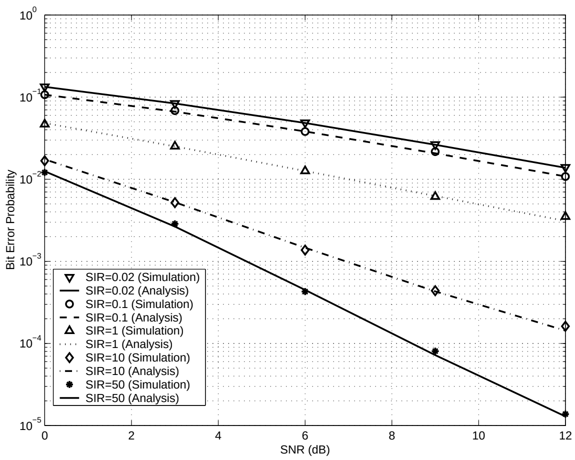

Fig. 2: BEP versus SNR for $N = 4$ branches, $L = 3$ Gaussian distributed interferers.

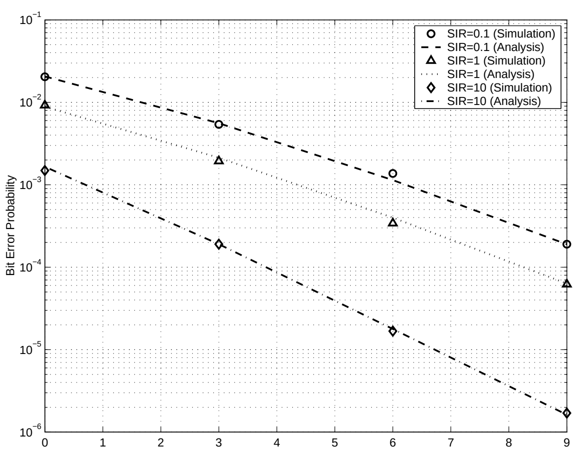

Fig. 3: BEP versus SNR for $N = 8$ branches, $L = 5$ Gaussian distributed interferers.

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