Performance Analysis of Reduced-Rank STAP

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Abstract– The space-time radar problem is well suited to the application of techniques that take advantage of the low-rank property of the space-time covariance matrix. In particular, it was shown that when the space-time covariance matrix is estimated from a dataset with limited support, reduced-rank methods outperform full-rank space-time adaptive processing (STAP). In this paper we study the application of several reduced-rank methods to the STAP problem and demonstrate their utility by simulations in terms of the output signal-to-noise ratio and detection probability. It is shown that reduced-rank processing has two opposite effects on the performance: increased statistical stability which tends to improve performance, and introduction of a bias which lowers the signal-to-noise ratio. Several reduced-rank methods are analyzed and compared for both cases of known and unknown covariance matrix. While best performance is obtained using transforms based on the eigendecomposition (data dependent), the loss incurred by the application of fixed transforms (such as the discrete cosine transform) is relatively small. The main advantage of fixed transforms is the availability of efficient computational procedures for their implementation. These findings suggest that reduced-rank methods could facilitate the development of practical, real-time STAP technology.

I. Introduction

Space-time adaptive processing (STAP) radar performance has been for some time a topic of considerable interest to the radar community. The optimal Neyman-Pearson detector for a known signal vector in colored Gaussian noise with a known covariance matrix is linear, i.e., it constructed from a linear combination of the vector's components. In practice, the interference+noise covariance matrix is typically not known. The common approach is to estimate it from a *secondary* data set (a set that excludes the cell-under-test). Reed et al. suggested the sample matrix inversion (SMI) method, in which an estimate is substituted for the noise covariance matrix expression in the linear detector [1]. They developed an expression for the density of the SNR loss with respect to the optimal case and showed that if the signal vector dimension is N, the number of samples required to achieve performance within 3 dB of the optimal, is approximately $K \approx 2N$. While preserving the linear architecture of the detector, the SMI detector has a number of drawbacks: (1) it is not always optimal for detection performance when the covariance matrix is not known and is estimated from the data, (2) it requires a large sample support N, and (3) it is sensitive to calibration errors. Recent publications have shown the advantages of various forms of reduced-rank methods over the full-rank SMI method [2], [3], [4]. The sample support required by reduced-rank processing is only $K \approx 2r$, where r is the rank of the interference to be rejected. Equivalently, for $r \ll N$ and for the *same* sample support, the SNR loss associated with reduced-rank methods is smaller than for the SMI.

In this work we review and compare several reduced-rank methods. First, we first formulate the reduced-rank minimum variance beamformer (RR-MVB), which utilizes the principal components of a specified matrix transformation. RR-MVB is equivalent to the reduced-rank generalized sidelobe canceler (GSC). Another class of reduced-rank methods are based on the cross-spectral metric (CSM) [5], [6]. Those are also presented in the GSC context. The last of the reduced-rank methods reviewed is the eigencanceler [3]. It is interesting to note that reduced-rank methods are generally evaluated by the error they produce with respect to full-rank adaptive processing. When the true covariance matrix is known, reduced-rank methods are suboptimal to the Wiener solution. However, our interest in those methods arises since it has been shown that when the interference is contained within a subspace of the signal space, and the interference+noise covariance matrix is estimated from a dataset with limited support, reduced-rank methods actually *outperform* full-rank adaptive processing. This is explained

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by the presence, in addition to thermal noise effects, of errors resulting from the estimation process. Reduced-rank processing suppresses estimation errors at the cost of a bias in the SNR. The net effect, however, is a significant performance improvement for cases when the interference may be modeled as low-rank. Reduced-rank methods are clearly important for STAP radar, where a large number of degrees of freedom may be available. For a uniform array and for fixed PRF, the space-time clutter covariance matrix is essentially low-rank due to the inherent oversampling nature of the STAP architecture. Hence, the space-time radar problem is well suited to the application of techniques that take advantage of the low-rank property.

This paper is organized as follows: Section II contains an analysis of reduced-rank processing when the covariance matrix is known. The case when the covariance matrix is estimated from the data, is analyzed in Section III. Numerical results are provided in Section IV.

II. Reduced-Rank Processing with Known Covariance

Consider a space-time array with ν antennas uniformly spaced and a κ coherent pulse interval (CPI). Data is collected from a 3D data cube which consists of the antenna elements, pulses of the CPI, and range returns. The processing goal is to determine whether a target is present at a range of interest. To that end, an $N = \nu \times \kappa$ snapshot consisting of the 2D data slice at the range of interest, is processed to yield a decision statistic, which subsequently is compared to a threshold. Processing is carried out in the N dimensional signal space resulting from stacking the snapshot column-wise. Under hypothesis $\mathbf{H}_0$, the $N \times 1$ received vector $\mathbf{x}$ consists only of clutter and noise contributions:

$$\mathbf{x} = \mathbf{c} + \mathbf{v}, \tag{1}$$

where $\mathbf{x}$ is assumed a zero-mean, circularly symmetric Gaussian random vector with covariance matrix $\mathbf{R}$. Under hypothesis $\mathbf{H}_1$, $\mathbf{x}$ is given by

$$\mathbf{x} = a\mathbf{s} + \mathbf{c} + \mathbf{v}, \tag{2}$$

where a is a zero-mean, circularly symmetric Gaussian random variable with variance σ_s^2.

The colored noise[1] covariance matrix given by $\mathbf{R} = \mathrm{E}\left[(\mathbf{c}+\mathbf{v})(\mathbf{c}+\mathbf{v})^H\right]$, where the superscript denotes transpose and complex conjugate, is usually not known, hence an estimate is used instead. The estimate is derived from range cells in the vicinity of the tested range cell and is termed *secondary data*. The secondary data consists of clutter returns and, possibly, other interferences, such as jammers. The presence of narrowband jammers does not alter the signal model as presented, thus we restrict our attention only to clutter signals. The assumption is that the secondary data $\mathbf{x}_k$, $k = 1, \ldots, K$, has the same statistical properties as the tested cell under hypothesis model $\mathbf{H}_0$. It is well known that the covariance matrix of a well calibrated, uniform array employing a fixed PRF is approximately low rank, i.e., it can be written

$$\mathbf{R} = \mathbf{Q}_1\mathbf{\Lambda}_1\mathbf{Q}_1^H + \sigma_v^2\mathbf{Q}_2\mathbf{Q}_2^H \tag{3}$$

where the diagonal of the $r \times r$ matrix $\mathbf{\Lambda}_1$ consists of the r principal (largest) eigenvalues of $\mathbf{R}$, the columns of $\mathbf{Q}_1$ are the corresponding eigenvectors, σ_v^2 is the variance of the white noise, and the columns of $\mathbf{Q}_2$ are the remaining eigenvectors of $\mathbf{R}$. The actual value of r is scenario dependent, but it is upper bounded by $r \leq \nu + \kappa - 1$, (see for example [3]).

This signal model suggests that STAP could benefit from the application of reduced-rank (RR) methods. Such methods incur a loss in the signal-to-noise ratio (SNR), but their main advantage lies in their statistical stability. In this paper, SNR is understood to mean signal-to-noise and interference ratio. We first consider RR methods when the covariance is known, and subsequently proceed to analyze RR performance with unknown covariance. The analysis is carried out in the frameworks of the MVB and the GSC processors.

A diagram of the reduced-rank MVB is shown in Figure 1. The full-rank MVB weight vector is obtained as a solution to the optimization problem:

$$\min\ \mathbf{w}^H\mathbf{R}\mathbf{w} \qquad \text{subject to} \qquad \mathbf{s}^H\mathbf{w} = 1, \tag{4}$$

where $\mathbf{R} = E\left[\mathbf{x}_k\mathbf{x}_k^H\right]$, $\mathbf{x}_k$ are snapshots of the secondary data, and $\mathbf{s}$ is the steering vector. With RR-MVB, the vector $\mathbf{x}_k$ is pre-processed by a full column rank $N \times r$ matrix transformation $\mathbf{T}$. The RR data is then the $r \times 1$ vector $\mathbf{z}_k = \mathbf{T}^H\mathbf{x}_k$, the RR covariance matrix is $\mathbf{T}^H\mathbf{R}\mathbf{T}$, and the RR steering vector is $\mathbf{t} = \mathbf{T}^H\mathbf{s}$. The RR-MVB weight vector is the solution to

$$\min\ \mathbf{w}^H\left(\mathbf{T}^H\mathbf{R}\mathbf{T}\right)\mathbf{w} \qquad \text{subject to} \qquad \left(\mathbf{T}^H\mathbf{s}\right)^H\mathbf{w} = 1, \tag{5}$$

and is given by the $r \times 1$ weight vector

$$\mathbf{w} = k\left(\mathbf{T}^H\mathbf{R}\mathbf{T}\right)^{-1}\mathbf{T}^H\mathbf{s}, \tag{6}$$

where $k = \left(\mathbf{s}^H\mathbf{T}\left(\mathbf{T}^H\mathbf{R}\mathbf{T}\right)^{-1}\mathbf{T}^H\mathbf{s}\right)^{-1}$. Based on this weight vector, it is easy to show that the optimal SNR, μ, is given by

$$\mu = \mathbf{s}^H\mathbf{T}\left(\mathbf{T}^H\mathbf{R}\mathbf{T}\right)^{-1}\mathbf{T}^H\mathbf{s}, \tag{7}$$

where to simplify notation, it is assumed that the target power $\sigma_s^2 = 1$.

[1] Colored noise refers to the aggregate of noise+interference+clutter.

The reduced-rank GSC is shown in Figure 2. From the figure it is observed that the output can be expressed

$$y = y_c(k) - y_a(k) = \mathbf{w}_c^H \mathbf{x}_k - \mathbf{w}_a^H \mathbf{U}^H \mathbf{A} \mathbf{x}_k, \tag{8}$$

where $\mathbf{w}_c$, the weight vector of the nonadaptive portion, is just the steering vector $\mathbf{w}_c = \mathbf{s}$, $\mathbf{w}_a$ is the adaptive weight, the matrix $\mathbf{U}$ is a full column rank transformation, and $\mathbf{A}$ is set such the MVB and GSC methods are equivalent. The weight vector $\mathbf{w}_a$ is found as the solution to the unconstrained optimization problem

$$\min_{\mathbf{w}_a} \left(\mathbf{s} - \mathbf{A}^H \mathbf{U} \mathbf{w}_a\right)^H \mathbf{R} \left(\mathbf{s} - \mathbf{A}^H \mathbf{U} \mathbf{w}_a\right). \tag{9}$$

The overall GSC weight vector is then given by

$$\mathbf{w} = \left(\mathbf{I}_N - \mathbf{A}^H \mathbf{U} \left(\mathbf{U}^H \mathbf{A} \mathbf{R} \mathbf{A}^H \mathbf{U}\right)^{-1} \mathbf{U}^H \mathbf{A} \mathbf{R}\right) \mathbf{s}, \tag{10}$$

where $\mathbf{I}_N$ is the N-dimensional identity matrix. In [7] it is shown that for the MVB and GSC methods to be equivalent, the following conditions need to be met: $(i)$the matrix $\mathbf{A}$ must block the look direction, $\mathbf{A}\mathbf{s} = \mathbf{0}$, $(ii)\mathbf{s}^H\mathbf{s} = 1$. Assuming that $\mathbf{A}$ has full column rank, and that $\mathbf{s}$ is the only vector in the null space of $\mathbf{A}$, the dimensions of $\mathbf{A}$ are $(N-1) \times N$. Consequently, the rank reducing matrix $\mathbf{U}$ is $(N-1) \times r$. Multiple linear constraints can be incorporated in $\mathbf{A}$ resulting in a null space of dimension equal to the number of constraints. The output SNR (when target power $\sigma_s^2 = 1$) is given by

$$\mu = \left(\mathbf{s}^H \mathbf{R} \mathbf{s} - \mathbf{s}^H \mathbf{R} \mathbf{A}^H \mathbf{U} \left(\mathbf{U}^H \mathbf{A} \mathbf{R} \mathbf{A}^H \mathbf{U}\right)^{-1} \mathbf{U}^H \mathbf{A} \mathbf{R} \mathbf{s}\right)^{-1} \tag{11}$$

Various choices of the rank reducing transformation $\mathbf{U}$ are now considered:

1. The goal is to maximize μ. In turn, this is achieved by maximizing the term $\eta = \mathbf{s}^H \mathbf{R} \mathbf{A}^H \mathbf{U} \left(\mathbf{U}^H \mathbf{A} \mathbf{R} \mathbf{A}^H \mathbf{U}\right)^{-1} \mathbf{U}^H \mathbf{A} \mathbf{R} \mathbf{s}$. For a given reduced rank r, η is optimized by a transformation $\mathbf{U}$ that meets the relation

$$\mathbf{A}^H \mathbf{U} = \mathbf{Q}_1, \tag{12}$$

where $\mathbf{Q}_1$ consists of the r principal eigenvectors of $\mathbf{R}$. Assuming that the $(N-1) \times N$ signal blocking matrix $\mathbf{A}$ has full column rank, the elements of $\mathbf{U}$ can be obtained from the solution of a least-squares problem. It is easy to show that with this choice of $\mathbf{U}$:

$$\begin{aligned} \mu &= \left(\mathbf{s}^H \mathbf{R} \mathbf{s} - \mathbf{s}^H \mathbf{Q}_1 \Lambda_1 \mathbf{Q}_1^H \mathbf{s}\right)^{-1} \\ &= \mathbf{s}^H \mathbf{Q}_2 \Lambda_2^{-1} \mathbf{Q}_2^H \mathbf{s}. \end{aligned} \tag{13}$$

2. To avoid the complication of a least-squares problem, let the matrix $\mathbf{U}$ be restricted to consist of r of the $(N-1)$ eigenvectors of $\mathbf{R}_a = \mathbf{A}\mathbf{R}\mathbf{A}$, where $\mathbf{R}_a = \overline{\mathbf{Q}}_1 \overline{\Lambda}_1 \overline{\mathbf{Q}}_1^H + \overline{\mathbf{Q}}_2 \overline{\Lambda}_2 \overline{\mathbf{Q}}_2^H$, and $rank(\overline{\mathbf{Q}}_1) = r$. A natural choice would be to let the rank reducing transformation consist of the r principal eigenvectors of $\mathbf{R}_a$, i.e., $\mathbf{U} = \overline{\mathbf{Q}}_1$. A less intuitive, but optimal approach is suggested in [5], [6]: construct $\mathbf{U}$ from the r eigenvectors of $\mathbf{R}_a$ that maximize the quantity

$$\frac{\left|\overline{\mathbf{q}}_i^H \mathbf{A} \mathbf{R} \mathbf{s}\right|^2}{\overline{\lambda}_i}, \tag{14}$$

where $\overline{\mathbf{q}}_i$, $\overline{\lambda}_i$ are respectively eigenvectors and eigenvalues of $\mathbf{R}_a$. In the references, this method is referred as the *cross-spectral metric* (CSM) method.

3. Principal components decomposition, such as considered above, is data dependent. Fixed, reduced-rank transformations can be constructed by selecting the principal components of the discrete Fourier transform (DFT) or the discrete cosine transform (DCT). The cross spectral metric can also be used in conjunction with these fixed transforms [5], [6].

Rank-reducing transformations are now evaluated in the MVB framework. Consider the SNR at the MVB output, as given by eq. (7). When the transformation $\mathbf{T}$ is unitary, it has no effect on the output SNR, and $\mu = \mathbf{s}^H \mathbf{R}^{-1} \mathbf{s} = \mu_{\max}$. For any $N \times r$ rank reducing transformation $\mathbf{T}$, $r < N$, $\mu = \mathbf{s}^H \mathbf{T} \left(\mathbf{T}^H \mathbf{R} \mathbf{T}\right)^{-1} \mathbf{T}^H \mathbf{s} < \mu_{\max}$. Specific examples are considered below.

1. Consider how *Case 1* of the GSC translates to the MVB framework. By substituting eq. (12) in eq. (10), we obtain the equivalent MVB weight vector

$$\mathbf{w} = \left(\mathbf{I}_N - \mathbf{Q}_1 \mathbf{Q}_1^H\right) \mathbf{s}. \tag{15}$$

This relation establishes the equivalence between the reduced-rank GSC and the eigencanceler [3]. The eigencanceler is a method that produces the minimum norm weight vector meeting the set of linear constraints, and subject to the additional constraint of orthogonality to the interference subspace (formed by the principal eigenvectors of the space-time covariance matrix). Since $\mathbf{Q}_2 \mathbf{Q}_2^H = \mathbf{I}_N - \mathbf{Q}_1 \mathbf{Q}_1^H$, the eigencanceler is equivalent to the application of a rank reducing transformation $\mathbf{T} = \mathbf{Q}_2$, where the columns of $\mathbf{Q}_2$ span the noise subspace of the covariance $\mathbf{R}$. Indeed, eq. (15) is obtained by using $\mathbf{T} = \mathbf{Q}_2$ and eq. (3) in eq. (6). The eigencanceler requires that the partition of eigenvectors into $\mathbf{Q}_1$ and $\mathbf{Q}_2$ be such that the desired signal lies mostly in the noise subspace (this is indeed the case when $\mathbf{R}$ is estimated from the secondary data). The output SNR is given by

$$\mu = \mathbf{s}^H \mathbf{Q}_2 \left(\mathbf{Q}_2^H \mathbf{R} \mathbf{Q}_2\right)^{-1} \mathbf{Q}_2^H \mathbf{s}$$

$$
\begin{aligned}
&= \mathbf{s}^H \mathbf{Q}_2 \mathbf{\Lambda}_2^{-1} \mathbf{Q}_2^H \mathbf{s} \\
&< \mu_{\max}, \qquad (16)
\end{aligned}
$$

where

$$\mu_{\max} = \mathbf{s}^H \mathbf{Q}_1 \mathbf{\Lambda}_1^{-1} \mathbf{Q}_1^H \mathbf{s} + \mathbf{s}^H \mathbf{Q}_2 \mathbf{\Lambda}_2^{-1} \mathbf{Q}_2^H \mathbf{s}. \qquad (17)$$

2. The $N \times (r+1)$ matrix is given by $\mathbf{T} = [\mathbf{Q}_1, \mathbf{q}_{r+1}]$, i.e., it consists of the r principal components of the signal-plus-interference subspace, augmented with one of the eigenvectors of the noise subspace [4]. In this case $\mathbf{T}$ consists of the principal components of the Karhunen-Loeve transform. The output SNR is given by $\mu = \mathbf{s}^H \mathbf{Q}_1 \mathbf{\Lambda}_1^{-1} \mathbf{Q}_1^H \mathbf{s} + \mathbf{s}^H \mathbf{q}_{r+1} \lambda_{r+1}^{-1} \mathbf{q}_{r+1}^H \mathbf{s}$. Note that if the look direction is in the noise subspace of the transform $\mathbf{T}$, i.e., $\mathbf{T}^H \mathbf{s} = \mathbf{0}$, there is no solution that meets the linear constraint in eq. (5). This problem is circumvented in [8] by the augmentation of $\mathbf{T}$ with the vector $\mathbf{s}$, $[\mathbf{T}, \mathbf{s}] \rightarrow \mathbf{T}$.
3. Similar to *Case 3* of the GSC, the rank reducing transformation $\mathbf{T}$ may be constructed from the principal components of a fixed transform such as the DFT or the DCT.
4. The columns of $\mathbf{T}$ may be designed using the cross-spectral metric approach.

In conclusion of this section, when the covariance matrix $\mathbf{R}$ is known, a rank reducing transformation induces a loss in the SNR. In the next section, this loss will be incorporated in the performance evaluation of the case when the covariance is not known and is evaluated from the data.

III. Reduced-Rank Processing with Unknown Covariance

In practice, the space-time covariance matrix is not known and it needs to be estimated from the secondary data, as explained in the previous section. This is the application for which reduced rank methods are advantageous due to their improved statistical stability. Let the number of snapshots from the secondary dataset be equal to K. Then the estimated covariance matrix is given by $\widehat{\mathbf{R}} = \frac{1}{K} \sum_{k=1}^{K} \mathbf{x}_k \mathbf{x}_k^H$. In this section, the performance of reduced-rank processors is analyzed as a function of the sample support K.

A widely accepted measure of performance for radar systems is the probability of detection. In adaptive radar, detection probability is a function of the weight vector. Likewise, the weight vector is derived from estimates of the covariance matrix of the secondary data, and as such, its elements are random variables. This makes the detection probability realization-dependent. Insight into what affects performance can be obtained by expressing detection probability as a function of the *conditioned SNR* (CSNR). The CSNR is defined as the effective SNR obtained by the application of a particular method, normalized by the optimal SNR (known covariance matrix, full-rank case):

$$\rho = \frac{\mathrm{SNR_{eff}}}{\mathrm{SNR_{opt}}}. \qquad (18)$$

The conditioned SNR is a random variable, always bounded $0 \le \rho \le 1$. With SMI, the full-rank MVB weight vector is given by $\mathbf{w} = \widehat{\mathbf{R}}^{-1}\mathbf{s}$. It follows that the CSNR can be expressed

$$\rho = \frac{\left(\mathbf{s}^H \widehat{\mathbf{R}}^{-1} \mathbf{s}\right)^2}{\mathbf{s}^H \widehat{\mathbf{R}}^{-1} \mathbf{R} \widehat{\mathbf{R}}^{-1} \mathbf{s}} \frac{1}{\mathbf{s}^H \mathbf{R}^{-1} \mathbf{s}}. \qquad (19)$$

The density of the conditioned SNR for the SMI method with Gaussian data has been characterized in [1], and is given by the beta distribution with parameters K and N,

$$f(\rho) = B(N) = C (1-\rho)^{N-2} \rho^{K+1-N}, \qquad (20)$$

where $C = \Gamma(K+1) / (\Gamma(N-1)\Gamma(K+2-N))$, and $\Gamma(K+1) = K!$ is the standard gamma function. The notation $B(N)$ emphasizes the signal space parameter N of the beta distribution shown above. When a reduced-rank transformation $\mathbf{T}$ is applied to the data, the CSNR can be written:

$$\rho = \frac{\left(\mathbf{t}^H \widehat{\mathbf{\Sigma}}^{-1} \mathbf{t}\right)^2}{\mathbf{t}^H \widehat{\mathbf{\Sigma}}^{-1} \mathbf{\Sigma} \widehat{\mathbf{\Sigma}}^{-1} \mathbf{t}} \frac{1}{\mathbf{s}^H \mathbf{R}^{-1} \mathbf{s}}, \qquad (21)$$

where $\mathbf{t} = \mathbf{T}^H \mathbf{s}$ is an $r \times 1$ vector, and $\mathbf{\Sigma} = \mathbf{T}^H \mathbf{R} \mathbf{T}$, $\widehat{\mathbf{\Sigma}} = \mathbf{T}^H \widehat{\mathbf{R}} \mathbf{T}$ are $r \times r$ matrices. Of interest is to determine the distribution of ρ for various reduced-rank methods. It is important to distinguish between two cases: (1) the transformation $\mathbf{T}$ is fixed, (2) the transformation $\mathbf{T}$ is data dependent. The former case applies when $\mathbf{T}$ is formed from the DFT or the DCT. The transformation $\mathbf{T}$ is also fixed when it is formed by eigenvectors of the true covariance $\mathbf{R}$, but this case is of no great practical value as the assumption here is that $\mathbf{R}$ is not known. Rather, $\mathbf{T}$ is formed from the eigenvectors of the estimated covariance $\widehat{\mathbf{R}}$ and, as such, is data dependent.

When $\mathbf{T}$ is fixed, eq. (21) can be rewritten as follows:

$$\rho = \rho_r \rho_b, \qquad (22)$$

where ρ_r is the reduced-rank CSNR,

$$\rho_r = \frac{\left(\mathbf{t}^H \widehat{\mathbf{\Sigma}}^{-1} \mathbf{t}\right)^2}{\mathbf{t}^H \widehat{\mathbf{\Sigma}}^{-1} \mathbf{\Sigma} \widehat{\mathbf{\Sigma}}^{-1} \mathbf{t}} \frac{1}{\mathbf{t}^H \mathbf{\Sigma}^{-1} \mathbf{t}} = B(r), \qquad (23)$$

and ρ_b is the bias in the optimal SNR introduced by the transformation $\mathbf{T}$,

$$\rho_b = \frac{\mathbf{t}^H \mathbf{\Sigma}^{-1} \mathbf{t}}{\mathbf{s}^H \mathbf{R}^{-1} \mathbf{s}}. \qquad (24)$$

Equations (22)-(24) clearly demonstrate the effect of reduced rank transformation on the SMI-MVB method. The linear transformation $\mathbf{T}$ preserves the Gaussian distribution of the data, hence the reduced-rank CSNR, ρ_r, has a beta distribution $B(r)$ with parameters K and r. Improved statistical stability is evident in the higher CSNR values associated with ρ_r. For example, for the full-rank SMI, $\mathbf{E}[\rho] = 0.5$ for $K = 2N - 3$ [1], while for the reduced-rank SMI, $\mathbf{E}[\rho_r] = 0.5$ for $K = 2r - 3$, i.e., fewer samples are required for the same performance level. The higher CSNR values due to improved statistical stability, are somewhat offset by the bias ρ_b, which is the loss in the optimal SNR due to the rank reduction. This loss is the quantity $\mu/\mu_{\max}$ analyzed in the previous section. The performance of the GSC processor with a fixed rank-reducing transformation is analyzed in [9].

The case when $\mathbf{T}$ is data dependent cannot be directly derived from the SMI distribution. The asymptotic density of the conditioned SNR for the eigencanceler ($\mathbf{T}$ consists of the noise eigenvectors of the estimated covariance) is derived in [10].

IV. Numerical Results

In this section are provided numerical results for several of the reduced-rank methods presented in previous sections. The simulation model used an $\nu = 8$ element array with $\kappa = 4$ taps at each element. The clutter was located in the angular sector of 0 to 30 degrees. The steering vector was set at 50 degrees, and at a normalized Doppler frequency of 0.4. The total input clutter-to-noise ratio (CNR) of the distributed clutter was 10 dB. In Figure 3, the distribution of ρ based on 20,000 runs is shown for several reduced-rank methods, as well as for full-rank SMI. The reduced-rank methods were: eigencanceler, DCT, DFT, and CSM based on the eigen-decomposition and implemented as a GSC. The number of principal components used to generate the results shown in the figure was $r = 4$ for all methods. The CSM and eigencanceler methods are shown to produce the highest CSNR's, with reduced-rank MVB based on the DCT and DFT providing slightly lower performance. All reduced-rank methods clearly outperform the full-rank SMI. Figure 4 plots the average probability of detection based on 200 runs and a false alarm probability of 10^{-5}. The figure illustrates the same trend as Figure 3; best detection performance is provided by the eigencanceler and CSM, followed by DCT and DFT (indistinguishable), and by the full-rank SMI. The effect of the rank order on the performance is illustrated in Figure 5. In the figure, the CSNR is plotted as a function of rank of the rank-reducing transformation. For all methods it seems that $r = 4$ is the optimal rank order (for the particular scenario considered). CSM provides slightly better performance when the rank is underestimated. The DCT transformation seems to be the least affected by overestimating the rank. Obviously, the rank has no effect on the SMI method. The effect of the CNR on performance is illustrated in Figure 6. The CSNR is plotted as a function of the input CNR. The CSNR is computed for a rank-reducing transformation with $r = 4$. As the CNR increases, the interference power spills over more than 3-4 principal values. Thus the rank reducing transformations are inadequate in capturing the interference power and performance is degraded.

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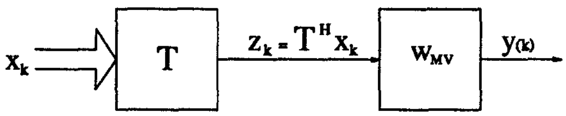

Fig. 1. Reduced-rank MVB.

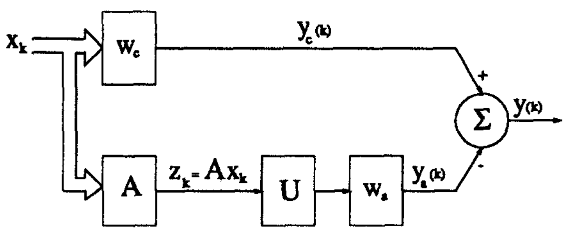

Fig. 2. Reduced-rank GSC.

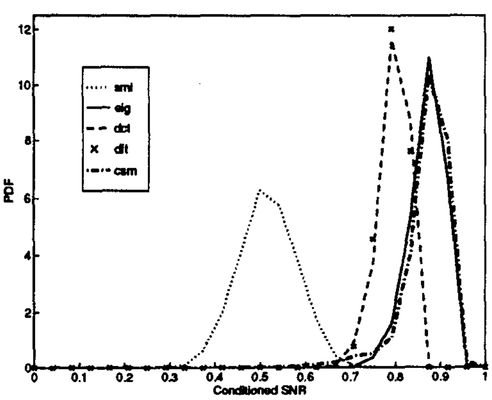

Fig. 3. Probability density of the CSNR.

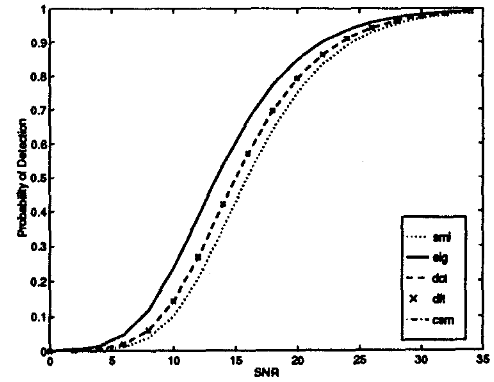

Fig. 4. Probability of detection for reduced-rank methods.

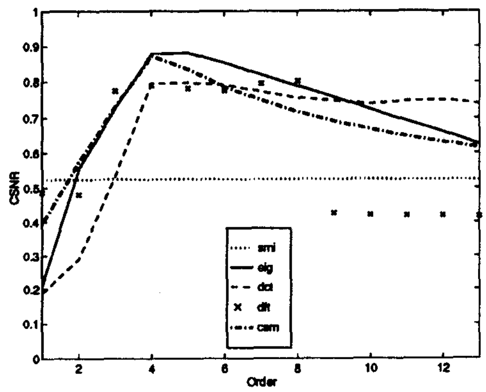

Fig. 5. Rank order.

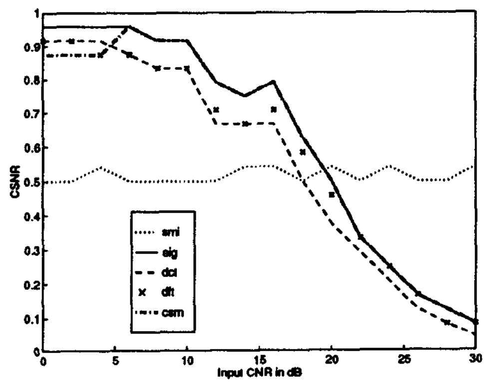

Fig. 6. CSNR vs CNR.