

# BER Analysis of MPSK Space-Time Block Codes With Differential Detection

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**Abstract**—Closed-form expressions of the bit error rate (BER) are derived for space-time block codes based on Alamouti's scheme and utilizing  $M$ -ary phase shift keying modulation with noncoherent differential encoding/decoding. The analysis is carried out for the flat, block-fading Rayleigh channel, and the BER expression is an approximation for high signal to noise ratio. Theoretical results are validated by simulations for BPSK and QPSK modulations.

**Index Terms**—Bit error rate (BER), differential detection, space-time block code (STBC).

## I. INTRODUCTION

THIS letter is motivated by the observation that for the special case of space-time block codes (STBCs) based on Alamouti's scheme [1], it is possible to obtain a closed-form expression for the bit error rate (BER). A closed-form BER expression would serve as an attractive alternative to previously derived bounds for evaluating performance of STBC [2], [3]. It is widely understood that Alamouti's scheme has a performance loss of 3 dB with respect to two-branch maximum ratio combining (MRC). It follows that the BER for Alamouti's scheme with coherent detection can be obtained directly from the result on receive diversity [4]. Here, we are concerned with the BER analysis for STBC with (noncoherent) differential detection. While the procedure for deriving the BER applies to any  $M$ -ary phase shift keying (MPSK) signaling, binary PSK (BPSK) and quadrature PSK (QPSK) examples are worked out in detail.

## II. SYSTEM MODEL

Assume a communication system with two transmit antennas and  $Q$  receive antennas operating over a flat, Rayleigh channel. The transmission scheme generates blocks of two symbols transmitted by each of the two antennas. With each block, we associate a time index  $k = 1, \dots, K$ . The received signal at time index  $k$ , time slots 1, 2 within the block, and receive antenna  $q$ ,  $1 \leq q \leq Q$ , is given by

$$\begin{bmatrix} r_{1,k}^{(q)} \\ r_{2,k}^{(q)} \end{bmatrix} = \begin{bmatrix} c_k^{(1)} & c_k^{(2)} \\ -c_k^{(2)*} & c_k^{(1)*} \end{bmatrix} \begin{bmatrix} h_1^{(q)} \\ h_2^{(q)} \end{bmatrix} + \begin{bmatrix} n_{1,k}^{(q)} \\ n_{2,k}^{(q)} \end{bmatrix} \quad (1)$$

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where  $h_1^{(q)}$ ,  $h_2^{(q)}$  are the path gains from transmit antenna 1 and 2, respectively to receive antenna  $q$ . Path gains are modeled as complex Gaussian random variables with zero-mean and variance  $1/2$  per dimension. The channel is assumed block-fading, i.e., fixed over the duration of  $K$  blocks. Noise samples  $n_{1,k}^{(q)}$ ,  $n_{2,k}^{(q)}$  are zero-mean complex Gaussian random variables with variance  $N_0/2$  per dimension. The symbols  $c_k^{(i)}$  are differentially encoded, transmitted by antennas  $i = 1, 2$ . Without loss of generality, we assume that the symbols amplitude is  $1/\sqrt{2}$  (such that the signal to noise ratio per symbol is  $1/2N_0$ ). The superscript '\*' denotes complex conjugation.

The differential STBC scheme analyzed in this letter is the one recently proposed by Tarokh and Jafarkhani [5] based on Alamouti's transmit diversity scheme. The message matrix is represented by the unitary  $2 \times 2$  matrix

$$\mathbf{S}_k = \begin{bmatrix} s_k^{(1)} & s_k^{(2)} \\ -s_k^{(2)*} & s_k^{(1)*} \end{bmatrix} \quad (2)$$

where the symbols  $s_k^{(i)}$  belong to an MPSK constellation. The message matrix  $\mathbf{S}_k$  is differentially encoded by a procedure resembling standard single-antenna DPSK. To initialize transmission, the transmitter sends a code unitary matrix  $\mathbf{C}_0$ , for example

$$\mathbf{C}_0 = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}. \quad (3)$$

The differentially encoded message  $\mathbf{C}_k$  at time  $k$ ,  $k \geq 1$ , is obtained by multiplying the codeword at time  $k-1$ ,  $\mathbf{C}_{k-1}$  by the current message  $\mathbf{S}_k$ , namely  $\mathbf{C}_k = \mathbf{S}_k \mathbf{C}_{k-1}$ . Note that the codeword  $\mathbf{C}_k$  has the same unitary property as the message matrix  $\mathbf{S}_k$ . In the absence of noise, the messages  $\mathbf{S}_k$  can be decoded from  $\mathbf{C}_k \mathbf{C}_{k-1}^\dagger = \mathbf{S}_k \mathbf{C}_{k-1} \mathbf{C}_{k-1}^\dagger = \mathbf{S}_k$ , where ' $\dagger$ ' denotes the Hermitian operation.

At the receiver, at the output of each antenna, we form the matrices  $\mathbf{R}_k^{(q)}$ ,  $1 \leq q \leq Q$ ,

$$\mathbf{R}_k^{(q)} = \mathbf{C}_k \mathbf{H}^{(q)} + \mathbf{N}_k^{(q)} \quad (4)$$

where

$$\mathbf{R}_k^{(q)} = \begin{bmatrix} r_{1,k}^{(q)} & -r_{2,k}^{(q)*} \\ r_{2,k}^{(q)} & r_{1,k}^{(q)*} \end{bmatrix}, \quad \mathbf{C}_k = \begin{bmatrix} c_k^{(1)} & c_k^{(2)} \\ -c_k^{(2)*} & c_k^{(1)*} \end{bmatrix},$$

$$\mathbf{H}^{(q)} = \begin{bmatrix} h_1^{(q)} & -h_2^{(q)*} \\ h_2^{(q)} & h_1^{(q)*} \end{bmatrix}, \quad \mathbf{N}_k^{(q)} = \begin{bmatrix} n_{1,k}^{(q)} & -n_{2,k}^{(q)*} \\ n_{2,k}^{(q)} & n_{1,k}^{(q)*} \end{bmatrix}.$$

Note that this construction ensures that  $\mathbf{R}_k^{(q)} \mathbf{R}_k^{(q)\dagger}$  is a diagonal matrix with equal entries. For  $Q$  receive antennas, the signal model is

$$\mathbf{R}_k = \mathbf{D}_k \mathbf{H} + \mathbf{N}_k, \quad (5)$$

where

$$\begin{aligned}\mathbf{R}_k &= [\mathbf{R}_k^{(1)} \mathbf{R}_k^{(2)} \dots \mathbf{R}_k^{(Q)}], \\ \mathbf{N}_k &= [\mathbf{N}_k^{(1)} \mathbf{N}_k^{(2)} \dots \mathbf{N}_k^{(Q)}], \\ \mathbf{D}_k &= \mathbf{C}_k \otimes \mathbf{1}_Q, \\ \mathbf{H} &= \text{diag} [\mathbf{H}^{(1)} \mathbf{H}^{(2)} \dots \mathbf{H}^{(Q)}]\end{aligned}$$

the symbol  $\otimes$  denotes the Kronecker product and  $\mathbf{1}_Q$  is a vector of ones of dimension indicated by the subscript. Stemming from the unitary property of  $\mathbf{C}_k$ , the matrices  $\mathbf{D}_k$  have the property  $\mathbf{D}_k \mathbf{D}_k^\dagger = Q \mathbf{I}_2$ , where  $\mathbf{I}$  denotes the identity matrix of dimension indicated by the subscript. Likewise,  $\mathbf{D}_k = \mathbf{S}_k \mathbf{D}_{k-1}$ . Finally, the channel model incorporating all  $K$  transmitted blocks can be put in the following matrix form:

$$\mathbf{R} = \mathbf{D} \mathbf{H} + \mathbf{N} \quad (6)$$

where  $\mathbf{R} = [\mathbf{R}_1 \mathbf{R}_2 \dots \mathbf{R}_K]^T$  is a  $2K \times 2Q$  matrix, the superscript denotes transposition, and both  $\mathbf{D}$  and  $\mathbf{N}$  are defined analogous to  $\mathbf{R}$ .

### III. BER ANALYSIS

It is known that the optimal maximum likelihood (ML) receiver with unknown channel state information is given by [6]:  $\hat{\ell} = \arg \max_{\ell} \text{tr} \left\{ \mathbf{R}^\dagger \mathbf{D}_\ell \mathbf{D}_\ell^\dagger \mathbf{R} \right\}$ , where  $\mathbf{D}_\ell$  represents a specific sequence of transmitted messages. The optimal receiver is based on the entire received sequence (6) and since its complexity is exponential in the sequence length  $K$ , a simpler suboptimal receiver is suggested based on only the last two received blocks. In this case, the quadratic detector reduces to:  $\hat{\ell} = \arg \max_{\ell} \text{Re} \text{tr} \left\{ \mathbf{S}^{(\ell)\dagger} \mathbf{R}_k \mathbf{R}_{k-1}^\dagger \right\}$ , where  $\mathbf{S}^{(\ell)}$  is a message matrix. The suboptimal receiver can be interpreted as a demodulator that generates the matrices  $\mathbf{G}_k = \mathbf{R}_k \mathbf{R}_{k-1}^\dagger$  followed by a decision mechanism. Consider now the properties of the demodulator output  $\mathbf{G}_k$ . Exploiting previous definitions, the (1, 1) element of this matrix can be expressed

$$\begin{aligned}g = \sum_{q=1}^Q \left\{ \left( |h_1^{(q)}|^2 + |h_2^{(q)}|^2 \right) s_k^{(1)} + h_1^{(q)} \left( c_k^{(1)} n_{1,k-1}^{(q)*} - c_{k-1}^{(2)*} n_{2,k}^{(q)*} \right) \right. \\ \left. + c_{k-1}^{(1)*} \nu_1^{(q)} n_{1,k}^{(q)} - c_k^{(2)} \nu_1^{(q)*} n_{2,k-1}^{(q)} \right) \\ \left. + h_2^{(q)} \left( c_k^{(2)} n_{1,k-1}^{(q)*} + c_{k-1}^{(1)*} n_{2,k}^{(q)*} \right) \right. \\ \left. + c_{k-1}^{(2)*} \nu_2^{(q)} n_{1,k}^{(q)} + c_k^{(1)} \nu_2^{(q)*} n_{2,k-1}^{(q)} \right\} \quad (7)\end{aligned}$$

where  $\nu_i^{(q)} = h_i^{(q)*} / h_i^{(q)}$ ,  $i = 1, 2$ , is a random variable and second order noise terms were neglected. It follows that the complex scalar  $g$  can serve as a decision statistic for the detection of the message symbol  $s_k^{(1)}$ . In (7), let

$$\begin{aligned}X_1^{(q)} &= h_1^{(q)}, \quad X_2^{(q)} = h_2^{(q)}, \\ Y_1^{(q)} &= h_1^{(q)*} s_k^{(1)} + c_k^{(1)} n_{1,k-1}^{(q)*} - c_{k-1}^{(2)*} n_{2,k}^{(q)*} \\ &\quad + c_{k-1}^{(1)*} \nu_1^{(q)} n_{1,k}^{(q)} - c_k^{(2)} \nu_1^{(q)*} n_{2,k-1}^{(q)}, \\ Y_2^{(q)} &= h_2^{(q)*} s_k^{(1)} + c_k^{(2)} n_{1,k-1}^{(q)*} + c_{k-1}^{(1)*} n_{2,k}^{(q)*} \\ &\quad + c_{k-1}^{(2)*} \nu_2^{(q)} n_{1,k}^{(q)} + c_k^{(1)} \nu_2^{(q)*} n_{2,k-1}^{(q)}.\end{aligned} \quad (8)$$

Then  $g$  can be expressed as  $g = \sum_{q=1}^Q \left( X_1^{(q)} Y_1^{(q)} + X_2^{(q)} Y_2^{(q)} \right)$ . Consider the properties of the random variables defined in (8). Under the assumptions delineated in the Section II, all these random variables have zero-mean. Moreover,  $X_1^{(q)}$ ,  $X_2^{(q)}$  are independent and identically distributed (i.i.d.) complex-valued Gaussian with unity variance. The variates  $Y_1^{(q)}$ ,  $Y_2^{(q)}$  require a bit more careful consideration due to their dependence on  $\nu_1^{(q)}$  and  $\nu_2^{(q)}$ , respectively. It is not difficult to show that  $\nu_i^{(q)}$  are of the form  $\nu_i^{(q)} = \exp(-j2\phi)$ , where  $\phi$  is a uniformly distributed phase in the interval  $(-\pi/2, \pi/2)$ . It follows that multiplication by  $\nu_i^{(q)}$  does not affect the distribution of complex Gaussian variates in (8). It is easy to see that  $E \left[ |Y_1^{(q)}|^2 \right] = E \left[ |Y_2^{(q)}|^2 \right] = 1/2 + 2N_0$ . The pairs  $(X_i^{(q)}, Y_i^{(q)})$ ,  $i = 1, 2$ ,  $q = 1, \dots, Q$ , are correlated, complex-valued, zero-mean Gaussian. Moreover, it can be shown that  $E \left[ Y_1^{(q)} Y_2^{(q)*} \right] = 0$ , but details are omitted here due to space considerations. Based on these observations, the signal model and (8), we conclude that the pairs  $(X_i^{(q)}, Y_i^{(q)})$  are i.i.d. This conclusion facilitates the application of known results for the development of BER analysis.

Since  $s_k^{(1)}$  is an MPSK symbol, the phase of  $g$  can serve as the decision variable:  $\theta = \tan^{-1}(\text{Im}(g)/\text{Re}(g))$ . For BER analysis, it is necessary to evaluate the probability distribution of the decision statistic  $\theta$ . To proceed with the BER computation, and without loss of generality, assume symbol  $s_k^{(1)}$  has zero phase, i.e.,  $s_k^{(1)} = 1/\sqrt{2}$ . Our problem closely matches the one solved in [4, App. C]. The probability distribution of the random variable  $\theta$  is given by

$$\begin{aligned}\Pr(\theta_1 \leq \theta \leq \theta_2) \\ = -\frac{(-1)^{2Q-1} (1-\mu^2)^{2Q}}{2\pi(2Q-1)!} \frac{\partial^{2Q-1}}{\partial b^{2Q-1}} [f(b, \alpha_2) - f(b, \alpha_1)]|_{b=1}\end{aligned} \quad (9)$$

where the function  $f(b, \alpha)$  is defined as follows:

$$\begin{aligned}f(b, \alpha_i) = \frac{1}{b - \mu^2} \left[ \frac{\mu \sqrt{1 - \left(\frac{b}{\mu^2 - 1}\right) \alpha_i^2}}{b^{1/2}} \cot^{-1} \alpha_i \right. \\ \left. - \cot^{-1} \left( \frac{\alpha_i b^{1/2}}{\mu \sqrt{1 - \left(\frac{b}{\mu^2 - 1}\right) \alpha_i^2}} \right) \right] \quad (10)\end{aligned}$$

$$\alpha_i = \frac{-\mu \cos \theta_i}{\sqrt{b - \mu^2 \cos^2 \theta_i}}, \quad i = 1, 2. \quad (11)$$

The term  $\mu$  represents the normalized cross-correlation between  $X_i^{(q)}$  and  $Y_i^{(q)}$  for  $i = 1, 2$  and  $q = 1, 2, \dots, Q$ . The normalized cross-correlation is defined as  $\mu = m_{xy} / \sqrt{m_{xx} m_{yy}}$ , where from (8),  $m_{xx} = E \left[ |X_i^{(q)}|^2 \right] = 1$  since by assumption  $E \left[ |h_i^{(q)}|^2 \right] = 1$ . Also, from (8),  $m_{yy} = 1/2 + 2N_0$ . Finally,  $m_{xy} = E \left[ X_i^{(q)} Y_i^{(q)*} \right] = \sqrt{1/2}$ . It follows that the normalized cross-correlation is given by  $\mu = \sqrt{\rho / (\rho + 2)}$ , where

$\rho = 1/2N_0$  is the signal-to-noise ratio (SNR) per symbol. Expression (9) can be used to evaluate the BER for specific MPSK modulations.

For BPSK signals, the BER can be obtained by integrating the density function of  $\theta$ ,  $p(\theta)$  over the ranges  $0.5\pi \leq \theta \leq \pi$  and  $1.5\pi \leq \theta \leq 2\pi$ . Since from [4, p. 891], the density  $p(\theta)$  is an even function of  $\theta$  (for any MPSK constellation), it follows that the BER is given by

$$P_2 = 2 \Pr \left( \frac{\pi}{2} \leq \theta \leq \pi \right). \quad (12)$$

Using (9) in (12) and after some algebraic manipulations, the closed-form BER for the noncoherent Alamouti scheme with BPSK is obtained.

$$P_2 = \frac{1}{2} \left[ 1 - \sqrt{\frac{\rho}{\rho+2}} \sum_{q=0}^{2Q-1} \binom{2q}{q} \left( \frac{1}{2(\rho+2)} \right)^q \right]. \quad (13)$$

For large SNR  $\rho$ , it can be shown that this expression can be approximated as

$$P_2 \simeq \frac{2^{2Q} \Gamma(0.5 + 2Q)}{\sqrt{\pi} \Gamma(1 + 2Q)} \left( \frac{1}{\rho} \right)^{2Q} \quad (14)$$

where  $\Gamma(\cdot)$  is the Euler gamma function. This expression demonstrates the diversity gain of  $2Q$ .

A Gray code is used to map pairs of bits onto QPSK symbols. For a transmitted symbol  $s_k^{(1)}$ , it is clear that a single bit error is committed when the received phase is  $(1/4)\pi \leq \theta \leq (3/4)\pi$ , and a double bit error is committed when the received phase is  $(3/4)\pi \leq \theta \leq \pi$ . Thus, the BER is expressed as

$$P_4 = \Pr \left( \frac{\pi}{4} \leq \theta \leq \frac{3\pi}{4} \right) + 2 \Pr \left( \frac{3\pi}{4} \leq \theta \leq \pi \right). \quad (15)$$

From (9) and (15), it can be shown that BER for STBC with QPSK modulation and differential encoding and decoding is given by

$$P_4 = \frac{1}{2} \left[ 1 - \sqrt{\frac{\rho}{\rho+4}} \sum_{q=0}^{2Q-1} \binom{2q}{q} \left( \frac{1}{\rho+4} \right)^q \right]. \quad (16)$$

#### IV. NUMERICAL RESULTS

Numerical results are provided to demonstrate the analysis developed in this letter and to compare it with simulation results. Fig. 1 shows the BER versus the SNR for BPSK STBC and Fig. 2 does the same for QPSK modulation. Curves for Alamouti's scheme with coherent detection are also shown for reference. The designation "2T1R" refers to a system with two transmit-one receive antennas. The slight bias at low SNR for the differential case is attributed to the second order noise terms, which were neglected in the analysis. The figures confirm that an approximately 3 dB performance gap exists between the coherent and differential schemes. Note that our performance for differential STBC with QPSK is a little better than [5] since bits are mapped to QPSK symbols using the Gray code.

#### V. CONCLUSION

We derived closed-form expressions of the BER over slow, flat Rayleigh fading for differential schemes based on Alam-

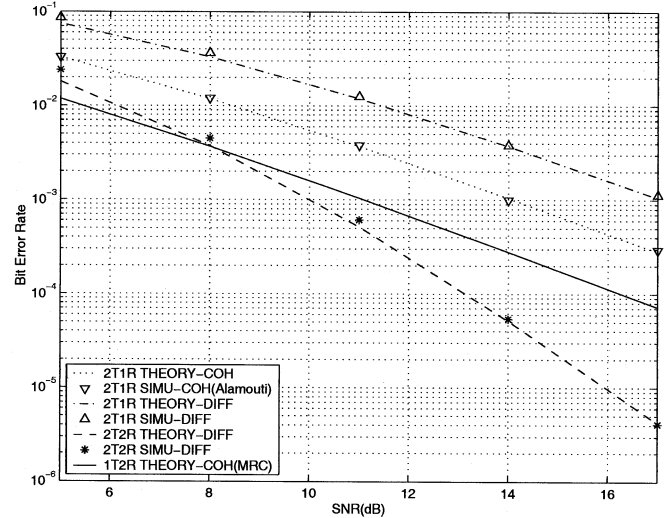


Fig. 1. Comparison of analysis and simulation with coherent and differential detection in BPSK case (1 bit/s/Hz).

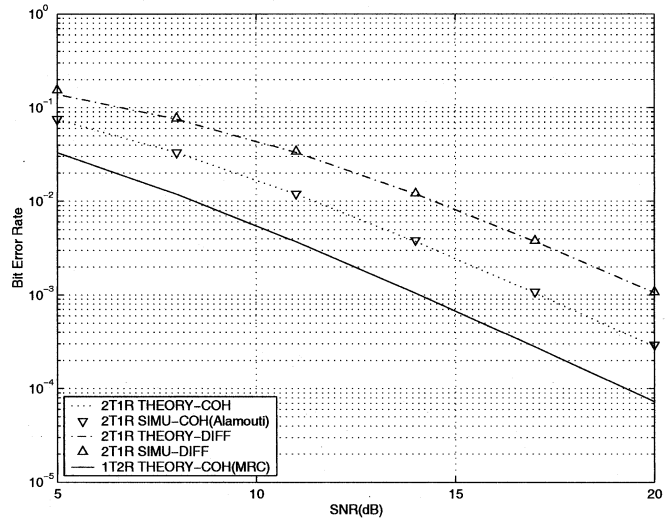


Fig. 2. Comparison of analysis and simulation with coherent and differential detection in QPSK case (2 bits/s/Hz).

outi's STBC. While the procedure outlined is applicable to any MPSK modulation, explicit BER expressions were obtained for BPSK and QPSK. Comparison of analytical and simulation results validates the new expressions.

#### REFERENCES

- [1] S. Alamouti, "A simple transmit diversity technique for wireless communications," *IEEE J. Select. Areas Commun.*, vol. 16, pp. 1451–1458, Oct. 1998.
- [2] X. Li, T. Luo, G. Yue, and C. Yin, "A squaring method to simplify the decoding of orthogonal space-time block codes," *IEEE Trans. Commun.*, vol. 49, pp. 1700–1703, Oct. 2001.
- [3] G. Bauch and J. Hagenauer, "Analytical evaluation of space-time transmit diversity with FEC coding," in *GLOBECOM '01*, vol. 1, 2001, pp. 435–439.
- [4] J. G. Proakis, *Digital Communications*, 3rd ed: McGraw-Hill, 1998.
- [5] V. Tarokh and H. Jafarkhani, "A differential detection scheme for transmit diversity," *IEEE J. Select. Areas Commun.*, vol. 18, pp. 1169–1174, July 2000.
- [6] B. Hughes, "Differential space-time modulation," *IEEE Trans. Inform. Theory*, vol. 46, pp. 2567–2578, Nov. 2000.