Error Performance Analysis of Multilevel Turbo-Code with Antenna Diversity

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Abstract— **A channel coding scheme is analyzed that features parallel concatenated systematic space-time codes with multilevel modulation and multiple transmit/receive antennas. The scheme is referred to as *turbo space-time coded modulation* (turbo-STCM). It combines the advantages of turbo codes with the diversity of space-time codes. The scheme features full diversity and full rate. In this paper, we also extend turbo-STCM to large number of antenna (> 2) cases. Simulation results are provided over the block-fading channel. It is shown that this turbo scheme provides an advantage of 1.1 dB over conventional space-time codes of similar decoding complexity. The analytical union bound of the bit error probability is derived for turbo-STCM over the Rayleigh block-fading channel. The bound makes it possible to express the performance analysis of turbo-STCM in terms of the properties of the constituent space-time codes. The union bound is demonstrated for 8-PSK turbo-STCM with two transmit antennas and one/two receive antennas.**

I. Introduction

Turbo-codes have been shown to provide excellent coding gains [1]. Space-time codes (STC) combine spatial diversity with the bandwidth efficiency and error correction coding of trellis coded modulation [2]. It can be expected that merging turbo and space-time codes could lead to the construction of better codes for the fading channel. Recently, several schemes that combine turbo and space-time codes were proposed. Serially concatenated space-time codes and turbo-codes were proposed in [3]. A serial concatenation of space-time codes and recursive convolutional codes was presented in [4]. A different serial concatenation as well as a parallel concatenation scheme were shown in [5]. In [6], the outputs of binary turbo-codes were mapped to QPSK symbols and transmitted through multiple antennas. A scheme that maps the outputs of recursive convolutional encoders to different antennas was proposed in [7]. Other parallel concatenations with the outputs of recursive encoders mapped to different antennas were presented in [8], [9]. Some of these schemes provide full space diversity and full coding rate.

A different approach is obtained by evolving from turbo-codes to trellis coded modulated turbo-codes [10]. Extending this idea, we proposed a scheme that features a parallel concatenation of two systematic space-time code modules [11], [12]. We referred to this scheme as turbo space-time coded modulation (turbo-STCM). It applies the principle of iterative processing directly to space-time codes. The output signals are punctured resulting in a full rate code. This scheme can be viewed as a true extension of the original turbo scheme [1] to multiple transmit antennas and multilevel modulation. In this paper, we extend the original two transmit antenna turbo-STCM scheme to a larger number of transmit antennas. In [13], we developed the upper bound for 2 bits/s/Hz 4-PSK turbo-STCM with two transmit antenna over the fading channel. In this paper, we provide the analysis for 3 bits/s/Hz 8-PSK turbo-STCM.

The paper is organized as follows. In the section II, we briefly present the system model and iterative decoding algorithm. An extension of turbo-STCM to large number of transmit antennas (> 2) is also introduced. Section III shows numerical simulations of turbo-STCM and comparisons with other methods. Section IV contains the theoretical performance analysis of turbo STCM. Numerical examples including simulations are found in section V. Finally, conclusions are presented in section VI.

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II. Turbo-STCM

The goal of turbo-STCM is to apply the turbo structure to space-time codes. The turbo-STCM encoder consists of two systematic space-time code modules operating in a parallel concatenation structure. A systematic recursive 16 state 8-PSK space-time code is shown in [14]. Systematic component codes are often encountered in turbo-codes applications due to their good performance at low signal-to-noise ratios. The systematic structure is further motivated by the need to puncture the parity data of the turbo code such the data rate of the overall code is the same as that of the constituent codes. A block diagram of the turbo-STCM encoder with two transmit antennas is shown in Fig. 1. Additional details on turbo-STCM encoding/decoding can be found in [11].

Turbo-STCM encodes m input information m bits to N symbols, to be transmitted by N transmit antennas. Using k to denote the time index, and for a turbo-STCM receiver utilizing an array of M antennas, the channel output symbols are represented by $M \times 1$ vectors

$$\mathbf{y}_k = \sqrt{E_s}\mathbf{H}\mathbf{s}_k + \mathbf{z}_k, \tag{1}$$

where E_s is the energy per symbol, related to the energy per bit by $E_s = mE_b$. The $M \times N$ matrix $\mathbf{H}$ consists of the channel coefficients. The channel is assumed flat, block-fading Rayleigh. Additive white Gaussian noise is modeled by the vector $\mathbf{z_k}$. The noise is assumed complex-valued, Gaussian distributed with zero-mean and variance $N_0/2$ for each dimension.

Turbo-STCM Extension

We present an extension of turbo-STCM scheme to a large number of transmit antenna ($N > 2$). The turbo coded symbols are divided into several substreams, and transmitted from several two-antenna groups simultaneously as shown in Fig. 2. In each group of two antennas, one antenna transmits a systematic symbol and the other antenna transmits the parity symbol. The output transmission rate is then $mN/2$ bits/s/Hz. At receiver, array processing techniques [15] are applied to extract each substream from the received signals. These substreams are then re-assembled into one whole sequence of observations, which corresponds to the transmitted codeword. Finally this sequence is fed into the turbo decoder.

Fig. 2 shows one example of turbo-STCM with a four transmit-three receive antennas configuration. The transmit antennas are separated into two groups. At receiver, to extract substreams from the multiple channel outputs by array processing, the number of receive antennas must satisfy $M \geq N - n_1 + 1$, where $n_1$is number of antennas in each group. In this example, $N = 4$, $n_1 = 2$, and $M = 3$. According to [15], the diversity gain of turbo-STCM scheme is $N \times (M + n_1 - N)$. In this example, the diversity gain is 4. A diversity loss occurs at the receiver since the antenna array is applied to interference cancellation rather than diversity. The payoff is in the higher data rate, which increases linearly with the number of transmit antennas, N. Furthermore, with the higher rate, the interleaver size of the turbo encoder can be increased without causing a longer delay.

III. Numerical Results

We present simulation results on the performance of our turbo-STCM scheme with different antenna setups for 2 and 3 bits/s/Hz. The block-fading channel model is understood to mean that the channel is assumed constant during a frame of $F = 130$ symbols, but independent frame-to-frame. Figure captions specify the number of transmit-receive antenna, e.g., 2T1R = two transmit-one receive antenna. Individual codes are labeled according to the modulation and number of states ("8p16s" refers to 8-PSK modulation, 16 state code).

Fig. 3 shows the frame error rate (FER) for two transmit-one receive antenna turbo-STCM, $K = 1300$ symbols interleaver and with recursive systematic 16 state 8-PSK constituent codes. Performance is shown for a fixed interleaver after 3 and 8 iterations. Curves are also provided for two transmit-one receive antenna 8-PSK 16 state and 64 state Tarokh *et al.* codes [2]. Finally, the performance of the recursive and systematic space-time constituent code is also shown. It is observed that at high E_b/N_0, all codes provide full diversity as demonstrated by the parallel asymptotic slopes. The systematic recursive code has the same diversity as Tarokh's code (same slope) but lower coding gain (approximately 1.5 dB at FER = 10^{-1} compared to Tarokh's 16 state code). However, after 3 iterations, turbo-STCM with fixed interleaver has an advantage of 2 dB over Tarokh's 16 state code at FER = 10^{-1}. This advantage becomes 3.4 dB after 8 iterations. The 64 state Tarokh code is shown since it has roughly the same decoding complexity as turbo-STCM with 16 state constituent codes, 3 iterations and two APP decoders per iteration. At FER = 10^{-1}, turbo-STCM has about 1.1 dB advantage after 3 iterations over a stand-alone space-time code of comparable complexity. This advantage becomes 2.3 dB after 8 iterations.

Fig. 4 shows the performance of four transmit-three receive antenna (4T3R) 4-PSK turbo-STCM and the comparison with a two transmit-one receive antenna (2T1R) setup. Curves are also provided for two transmit-one receive antenna 4-PSK 64 state Tarokh *et al.* codes [2]. It is observed that at FER=10^{-2}, the performance of the 4T3R turbo-STCM is around 1.5 dB better than the 2T1R turbo-STCM. The 4T3R turbo-STCM also outperforms Tarokh's STC 1.7 dB at FER=10^{-2}. Note that the encoding rate of the 4T3R turbo-STCM is 4 while in the 2T1R systems the rate is 2. From the curve slopes at high E_b/N_0, it is observed that the 4T3R turbo-STCM loses part of spatial diversity in order to increase the transmission rate.

IV. Turbo-STCM Performance Analysis

A method for computing the input-output weight enumerator (IOWE) coefficients $a(6K, w, z)$ for punctured STC is given in [11], [13]. Given a systematic block code $\mathbf{c}$ of length K symbols and the coefficients of the constituent STC codes, say $a_1(6K, w, z_1)$ and $a_2(6K, w, z_2)$, the turbo-STCM IOWE coefficients can be found from

$$\widetilde{a}(6K, w, z) = \sum_{z_1 + z_2 = z} \frac{a_1(6K, w, z_1) a_2(6K, w, z_2)}{\binom{3K}{w}}, \qquad (2)$$

where the Hamming weight w for the information bits and z for the parity bits, $0 \leq w \leq 3K$ and $0 \leq z \leq 3K$. We refer to such error sequences as (w, z) sequences. The overall Hamming weight of these sequences is $w + z$. This expression is used to evaluate the IOWE's of turbo-STCM.

2^m-PSK turbo-STCM accepts m (here $m = 3$) binary symbols at a time and transforms them into N (transmit antenna numbers) blocks of binary symbols that are fed into a memoryless mapper $\rho(\cdot)$. The noiseless received symbol is,

$$f(\mathbf{c}_k) = h_1 s_k^1 + h_2 s_k^2 = \mathbf{h}^T \rho(\mathbf{c}_k) \qquad (3)$$

where $\mathbf{h}^T = [h_1 h_2]$ is substituted for $\mathbf{H}$ in signal model (1). Let $\underline{\mathbf{c}}_K = (\mathbf{c}_1, \ldots, \mathbf{c}_K)$ be a sequence of K binary codewords associated with the K transmitted symbols. An error event occurs when the demodulation chooses transmitted symbols corresponding to $\underline{\mathbf{c}}_K \oplus \underline{\mathbf{e}}_K$ and $\underline{\mathbf{e}}_K$ is a non-zero sequence of binary error vectors.

The pairwise error probability of choosing $\underline{\mathbf{c}}_K \oplus \underline{\mathbf{e}}_K$ instead of $\underline{\mathbf{c}}_K$ is denoted $P(\underline{\mathbf{c}}_K \rightarrow \underline{\mathbf{c}}_K \oplus \underline{\mathbf{e}}_K)$ and is given by

$$P(\underline{\mathbf{c}}_K \rightarrow \underline{\mathbf{c}}_K \oplus \underline{\mathbf{e}}_K) = Q\left(\sqrt{\frac{r_c E_b d^2(\underline{\mathbf{c}}_K, \underline{\mathbf{e}}_K)}{2N_0}}\right) \leq \frac{1}{2} Z^{d^2(\underline{\mathbf{c}}_K, \underline{\mathbf{e}}_K)}, \quad (4)$$

where $Z = e^{-r_c E_b/4N_0}$, $r_c = m/Nm$ is the turbo-STCM code rate and $d^2(\underline{\mathbf{c}}_K, \underline{\mathbf{e}}_K) = \|f(\underline{\mathbf{c}}_K) - f(\underline{\mathbf{c}}_K \oplus \underline{\mathbf{e}}_K)\|^2$ is the cumulative squared Euclidean distance (SED) associated with $\underline{\mathbf{c}}_K$ and $\underline{\mathbf{e}}_K$.

The *type* of an error sequence is defined as the vector $\mathbf{n}$ with elements n_{ij} denoting the number of symbols in the error sequence that have i systematic bit errors and j parity bit errors. For the 3 bits/s/Hz, 8-PSK 2 transmit antennas we have $(i,j) \in$ (0,1),(0,2),(0,3),(1,0),(1,1),(1,2),(1,3),(2,0),(2,1),(2,2),(2,3),(3,0),(3,1),(3,2),(3,3). The following relations can be found

$$\begin{aligned} w &= n_{10} + n_{11} + n_{12} + n_{13} + 2n_{20} + 2n_{21} + 2n_{22} \\ &\quad + 2n_{23} + 3n_{30} + 3n_{31} + 3n_{32} + 3n_{33} \\ z &= n_{01} + 2n_{02} + 3n_{03} + n_{11} + 2n_{12} + 3n_{13} + n_{21} \\ &\quad + 2n_{22} + 3n_{23} + n_{31} + 2n_{32} + 3n_{33}. \end{aligned} \quad (5)$$

Following [11], the conditional upper bound on the bit error probability is given by

$$\begin{aligned} P(e|\mathbf{h}) &\leq B(e|\mathbf{h}), \\ &= \sum_{w=1}^{3K} \sum_{z=0}^{3K} \sum_{\mathbf{n}_\beta(w,z)} \frac{w}{6K} \tilde{a}(6K, w, z) \Phi(w, z, \mathbf{n}_\beta) E\left[Z^{D^2_{\mathbf{n}_\beta}}\right], \end{aligned} \quad (6)$$

where the term $E\left[Z^{D^2_{\mathbf{n}_\beta}}\right]$ is the expectation over SED's associated with error sequences of type $\mathbf{n}_\beta(\mathbf{w}, \mathbf{z})$ and the notation $B(e|\mathbf{h})$ denotes the bound. Then the average upper bound to the bit error probability is given by

$$B(e) = E_{\mathbf{h}}[B(e|\mathbf{h})] \quad (7)$$

$$= \sum_{w=1}^{3K} \sum_{z=0}^{3K} \sum_{\mathbf{n}_\beta(w,z)} \frac{w}{6K} \tilde{a}(6K, w, z) \Phi(w, z, \mathbf{n}_\beta) E_D\left\{E_{\mathbf{h}}[Z^{D^2_{\mathbf{n}_\beta}}]\right\},$$

where we switched the order of the expectation operations with respect to D and channel $\mathbf{h}$. The SED for an error path can be expressed

$$d^2(\mathbf{c_k}, \mathbf{e_k}) = \left\| \sum_{n=1}^{2} h_n(\rho(c_{n,k}) - \rho(c_{n,k} \oplus e_{n,k})) \right\|^2 \quad (8)$$

Subsequently, the cumulative SED associated with an error sequence can be expressed in terms of the channel:

$$\begin{aligned} d^2(\underline{\mathbf{c}}_K, \underline{\mathbf{e}}_K) &= \sum_{k=1}^{K} \left\| \sum_{n=1}^{2} h_n(\rho(c_{n,k}) - \rho(c_{n,k} \oplus e_{n,k})) \right\|^2 \\ &= \mathbf{x^H x} \end{aligned} \quad (9)$$

where the superscript H denotes complex and transposed. $\mathbf{x}$ is $K \times 1$ vector and has complex-valued Gaussian multivariate with zero mean and covariance matrix $\mathbf{M_x} = \mathrm{cov}(\mathbf{x}) = (\mathbf{m_{ij}})$.

$$(m_{ij}) = \begin{cases} i = j & \sum_{n=1}^{2} d^2(c_{n,i}, e_{n,i}) \\ i \neq j & 0 \end{cases}, \ 1 \leq i, j \leq K \quad (10)$$

From (4), (9) and the definition of Z, evaluation of

$$E_{\mathbf{h}}\left[Z^{D^2_{\mathbf{n}_\beta}} | D^2_{\mathbf{n}_\beta} = d^2(\underline{\mathbf{c}}_K, \underline{\mathbf{e}}_K)\right] = E_{\mathbf{h}}\left[e^{-\gamma \mathbf{x^H x}}\right], \quad (11)$$

where $\gamma = r_c E_b/4N_0$. Using the Laplace transform with respect to the quadratic form of a complex Gaussian random variable.

$$E_{\mathbf{h}}\left[e^{-\gamma \mathbf{x^H x}}\right] = \det(\mathbf{I_K} + \gamma \mathbf{M_x})^{-1} = \prod_{k=1}^{K} \left(1 + \gamma d_k^2\right)^{-1} \quad (12)$$

where $\mathbf{I_K}$ is the $K \times K$ identity matrix. The equivalent SED averaged over fading is

$$\prod_{k=1}^{K} \left(1 + \gamma d_k^2\right)^{-1} = \exp\left(-\sum_{k=1}^{K} \Delta_k^2\right), \quad (13)$$

The computation of the bound can now proceed as in the AWGN case, with the cumulative SED for a type $\mathbf{n}_\beta(\mathbf{w}, \mathbf{z})$ error sequence being represented by the random variable $\widetilde{D}^2_{\mathbf{n}_\beta}$, where $\widetilde{D}^2_{\mathbf{n}_\beta}$ has already been averaged over the fading channel. Thus the final expression for the bit error bound is

$$P_e \leq \sum_{w=1}^{3K} \sum_{z=0}^{3K} \sum_{\mathbf{n}_\beta(\mathbf{w},\mathbf{z})} \frac{w}{6K} \tilde{a}(6K, w, z) \Phi(w, z, \mathbf{n}_\beta) E\left[e^{\widetilde{D}^2_{\mathbf{n}_\beta}}\right] \quad (14)$$

The extension to multiple receive antenna is similar to [13].

V. Union Bound Examples

This part contains numerical examples that demonstrate the union bound derived in section IV. For comparison, simulation results are also provided. The simulation is based on Monte Carlo runs with 10,000 channel realizations for each E_b/N_0. As in the numerical examples of Section III, the channel model was Rayleigh block-fading, meaning that the channel was assumed constant during a frame of $F = 130$ symbols, but independent frame-to-frame. Simulation results shown are averaged over random interleavers of size as specified in the figures. The union bound is found for 16 state 8-PSK turbo-STCM with constituent codes shown in [14].

Fig. 5 and 6 respectively, provide the performance analysis for single and two receive antennas. Simulations results are shown after eight iterations of the turbo decoder. Each figure is divided in two parts corresponding to interleaver sizes K =1300 and 5200 symbols, respectively. The expression specifies a bound obtained by summing over all error sequences with systematic weights $1 \leq w \leq 3K$, and parity weights $0 \leq z \leq 3K$. Computation of the full bound is prohibitive in its complexity and it is also unnecessary. One of the analysis goals is to investigate how many terms are required to obtain a good approximation of the union bound. Each curve in each figure is labeled with the weights w of the error sequences. In all cases, the parity terms included in the bound were $0 \leq z \leq 15$. The justification of using a small subset of error sequences is that the small weights result in sequences with small SED's, which in turn, contribute the dominant terms to the union bound. From the figures it can be observed that error sequences corresponding to $w = 2$ are sufficient to provide a good approximation to the error bound. For example, it can be observed from Fig. 5 (b) that the curves for $w \geq 2$ coalesce for $E_b/N_0 \geq 20$ dB. Similarly, for the two receive antenna case, in Fig. 6 (b) the curves coalesce for $E_b/N_0 \geq 14$ dB. These results justify simplifying the bound evaluation by neglecting terms with large weights.

VI. Conclusion

In this paper we analyzed a turbo coding scheme, referred to as turbo-STCM, featuring space-time constituent codes. The scheme provides full diversity and full rate. Performance over a flat block-fading Rayleigh channel was demonstrated by simulations for 3 bits/s/Hz 16 state 8-PSK turbo-STCM codes. Turbo-STCM with a fixed interleaver has an performance improvement over 64 state Tarokh code of 2.3 dB after 8 iterations for the two transmit-one receive antenna configuration at FER $= 10^{-1}$. For large number of antenna configuration such as 4T3R turbo-STCM, the coding rate is doubled by sacrificing part of spatial diversity.

The analytical union bound was derived for 3 bits/s/Hz, 8-PSK turbo-STCM with two transmit-one/two receive antennas over the block-fading Rayleigh channel. The union bound is expressed in terms of the input output weight enumerators of the constituent codes, the interleaver size, and the spectrum of the squared Euclidean distance spectrum. It has been shown that a small number of terms with low systematic bit weights is sufficient to yield an upper bound of the bit error. The analysis developed in the paper provides a practical tool to analyze the performance of these turbo codes.

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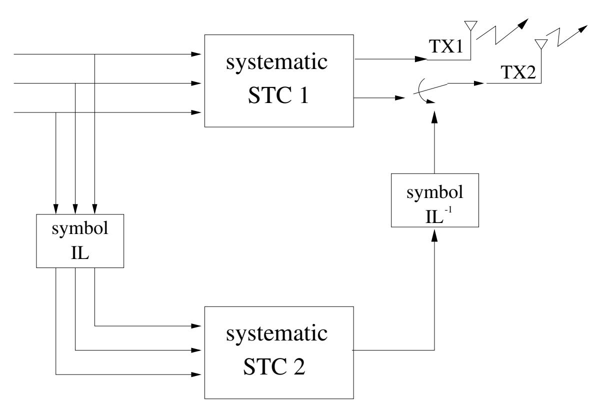

Fig. 1. Turbo-STCM encoder, 2 transmit antennas.

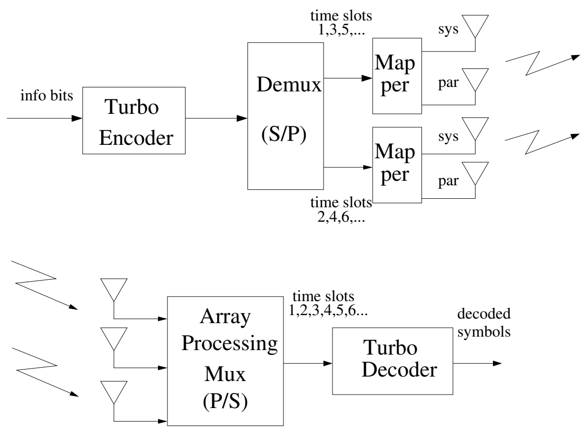

Fig. 2. Turbo coded multiple antenna scheme (4T3R).

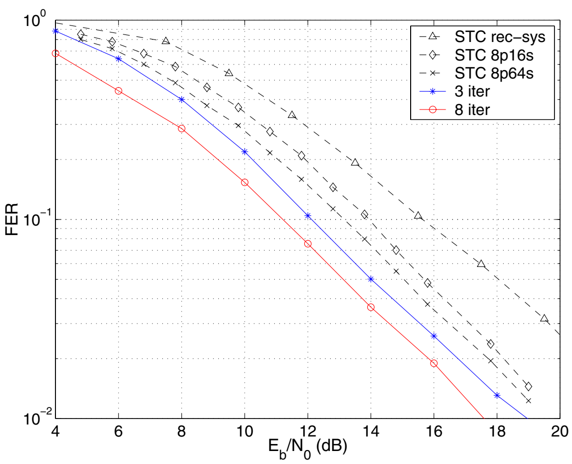

Fig. 3. FER of turbo-STCM (2T1R) 16 state 8-PSK with fixed interleaver $K = 1300$ symbols.

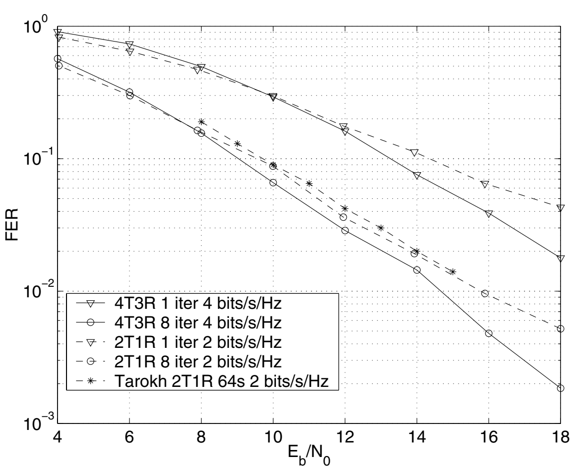

Fig. 4. FER of turbo-STCM (4T3R) 4 bits/s/Hz over fading channel and comparison with other methods.

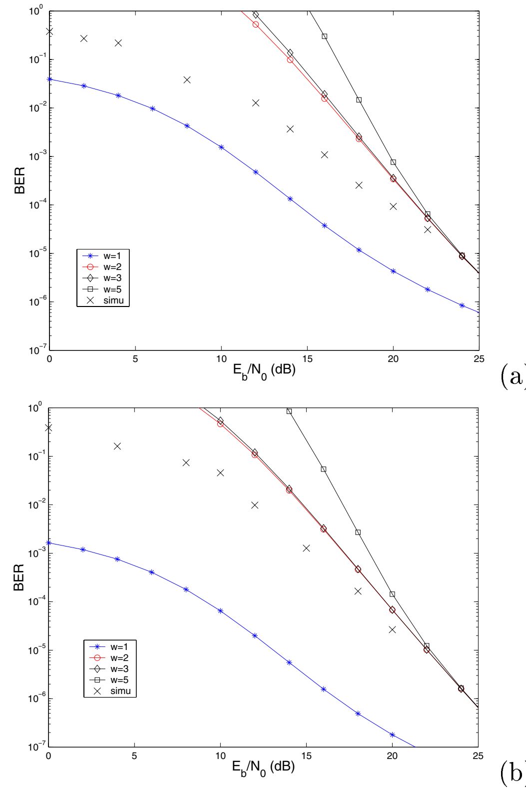

Fig. 5. Union bound for turbo-STCM (2T1R) over block-fading channel with block length (symbols): (a) $K = 1300$, (b) $K = 5200$.

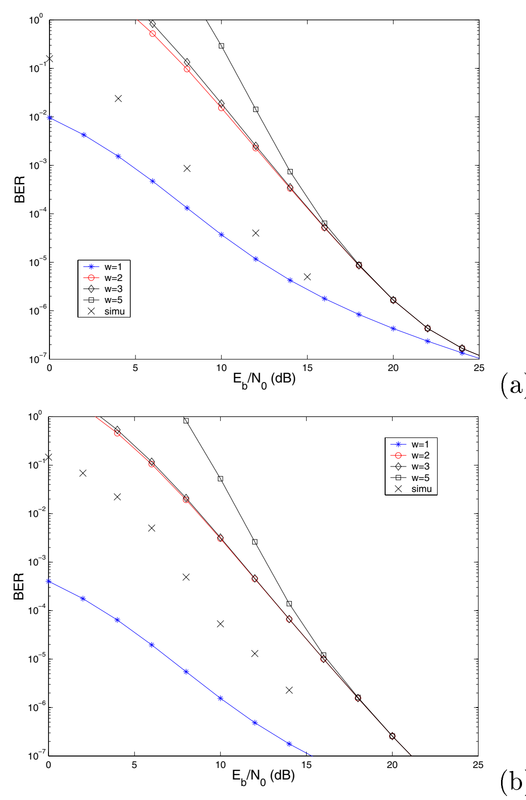

Fig. 6. Union bound for turbo-STCM (2T2R) over block-fading channel with block length (symbols): (a) $K = 1300$, (b) $K = 5200$.