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NINE

Miscellaneous Theorems

In the context of this book. several of the classical structural theorems take on an especially simple form and are collected in this chapter.

For all problems considered in this book. the system matrix has been symmetric. For example. in the node method

$$(\tilde{N}KN)\delta = P, \tag{9.1}$$

the system matrix $\tilde{N}KN$ is symmetric since in general

$$(\widetilde{ABC}) = \tilde{C}\tilde{B}\tilde{A}, \tag{9.2}$$

from which it follows that

$$(\widetilde{\tilde{N}KN}) = NKN. \tag{9.3}$$

if K is symmetric. (i.e. $\tilde{K} = K$). For all the cases considered. K has been symmetric. Rewriting Eq. (9.1) as

$$C\delta = P, \tag{9.4}$$

since C is symmetric C^{-1} is also symmetric. From these symmetries follow

MAXWELL'S LAW. *The displacement i due to a unit force j equals the displacement j due to a unit force i.* (There is a second form of this theorem which is obtained by interchanging the words force and displacement in the theorem stated.)

Proof: $C_{ij} = C_{ji}$.

As stated, this theorem is a little out of the spirit of this book in that when it refers to a displacement or force it means a single scalar force or displacement component.

Given two different sets of loads P^1 and P^2 and their associated displacements δ^1 and δ^2 produced on the same structure. To this situation applies

BETTI'S LAW. $\tilde{\delta}^1 P^2 = \tilde{\delta}^2 P^1$.

Proof: Eq. (9.4) applies to both loads to give

$$C\delta^1 = P^1 \quad \text{and} \quad C\delta^2 = P^2.$$

Multiplying by $\tilde{\delta}^2$ and $\tilde{\delta}^1$ respectively results in

$$\tilde{\delta}^2 C\delta^1 = \tilde{\delta}^2 P^1 \quad \text{and} \quad \tilde{\delta}^1 C\delta^2 = \tilde{\delta}^1 P^2$$

But $\tilde{\delta}^2 C\delta^1 = \tilde{\delta}^1 C\delta^2$, since C is symmetric and a scalar is equal to its transpose.

In Chapter 3 it was explained that the formulation of the node method implies that

$$W = \tfrac{1}{2}\tilde{P}\delta = \tfrac{1}{2}\tilde{F}\Delta = E, \tag{9.5}$$

from which it follows directly

CASTIGLIANO'S THEOREM. $\partial E/\partial(\delta)_i = (P)_i$.

Proof: $E = W = \tfrac{1}{2}\tilde{\delta}C\delta$. By direct differentiation it can be shown that $\partial E/\partial(\delta)_i = (C\delta)_i = (P)_i$.

Note that again there is a second form of the theorem which states that $\partial E/\partial(P)_i$ = (S), and which can also be proved. by direct differentiation.

VIRTUAL WORK. *Given any force system which satisfies equilibrium,*

$$\tilde{N}F^1 = P^1 \tag{9.6}$$

and any "conformable" set of displacements δ, *then*

$$\tilde{\Delta}F^1 = \tilde{\delta}P^1. \tag{9.7}$$

Proof: Eq. (9.7) follows directly by multiplying Eq. (9.6) on the left by $\tilde{\delta}$,

$$\tilde{\delta}\tilde{N}F^1 = \tilde{\delta}P^1. \tag{9.X}$$

and interpreting $N\delta$ = A as a set of member displacements. δ is "conformable" if N and δ are of proper dimensions so that $N\delta$ has meaning.

The generality of Eq. (9.7) lies in the fact that the displacements and forces do not have to occur in the same structure. This fact has been long recognized by engineers who have used statically determinate "virtual structures" when computing displacements in statically indeterminate structures.

Finally, geometric stability is easily discussed in the notation of this book. If a structure is (geometrically) unstable. there exists a displaced configuration $\delta \neq 0$ for which no bar forces aregenerated. The implication is that at least part of the structure is behaving like a mechanism under this displacement.

Let the matrix N have m columns.

THEOREM. *If a structure is unstable, the rank of KN must be less than m.*

Proof: By definition. if a structure is unstable. there exists a $\delta \neq 0$ and an associated $F = 0$.

Since

$$F = O = KA = KN\delta. \tag{9.9}$$

if the theorem were not true it would be possible to partition Eq. (9.9) after a re-arrangement of rows

$$\begin{bmatrix} F^1 \\ \hline F^2 \end{bmatrix} = \begin{bmatrix} (KN)^1 \\ \hline (KN)^2 \end{bmatrix}\delta.$$

so that $(KN)^1$ would have an inverse and F^1 would not in general be zero. This is the desired contradiction.

Note that in general stability depends upon K. In cases such as trusses in which none of the bar areas is zero and which therefore have non-singular primitive stiffness matrices. stability depends only upon the matrix N.

TEN

Kron's Methods

10.1 INTRODUCTION

With a little experience it becomes clear that one of the major problems facing the structural analyst is how to reduce the computational effort (cost) ofsolutions which is closely related to some very practical problems of describing structures and information retrieval. Large systems are addressed directly in Appendix A.5 but in this chapter some related work of the late Gabriel Kron is discussed as it applies to structures.

While the intention here is not to imply that Kron was the only worker in these fields, it seems clear now that he was far ahead of his time and, in fact, it still remains to investigate the generality and usefulness of some of his techniques.

10.2 TEARING AND INTERCONNECTING

Kron is best known for his method of "tearing and interconnecting" in which rather than solving a structure directly. pieces of it are first removed (the structure is torn) and the reduced structure is solved. Then the effect of the missing pieces is introduced (the structure is interconnected). Under the proper conditions. solution by tearing and interconnecting can produce a phenomenal cost saving.

Tearing and interconnection is most directly explained in terms of the method of modified matrices based upon the formula

$$(B - US\tilde{V})^{-1} = B^{-1} + B^{-1}U(S^{-1} - \tilde{V}B^{-1}U)^{-1}\tilde{V}B^{-1}, \qquad (10.1)$$

which can be verified directly by starting with the identity

$$US[(S^{-1} - \tilde{V}B^{-1}U)(S^{-1} - \tilde{V}B^{-1}U)^{-1} - I]\tilde{V}B = 0.$$

$(B + US\tilde{V})^{-1} = B^{-1} - B^{-1}U(S^{-1} + \tilde{V}B^{-1}U)^{-1}\tilde{V}B^{-1}$

adding I to each side of the equation and expanding.

$$I + U(S^{-1} - \tilde{V}B^{-1}U)^{-1}\tilde{V}B^{-1} - US\tilde{V}B^{-1} - US\tilde{V}B^{-1}U(S^{-1} - \tilde{V}B^{-1}U)^{-1}\tilde{V}B^{-1} = I,$$

producing finally

$$(B - US\tilde{V})\,[B^{-1} + B^{-1}U(S^{-1} - \tilde{V}B^{-1}U)^{-1}\tilde{V}B^{-1}] = I.$$

Equation (IO. I) fs usually applied to the situation in which the behavior of a system, B^{-1}, is known, given the system matrix B and it is desired to find the effect of a change in B, $-US\tilde{V}$, on the inverse system matrix B^{-1}.

The implications of Eq. (10.1) are quite broad and will be discussed here largely through examples, but it is first interesting to note the similarity in form between Eq. (IO. I) and Eq. (3.6).

Figure IO. I shows a very simple plane truss which is "torn" into two identical independent structures by removing bar 5. For simplicity. assume that for each bar in this problem $K_i = A_iE/L_i = $ I. The system matrix for the torn structure must have the form

$$(\tilde{N}KN)_{\text{torn}} = \begin{bmatrix} A & 0 \\ 0 & A \end{bmatrix}, \qquad (10.2)$$

since the two pieces are identical and independent.

Using the decomposition procedure described in Chapter 3. the matrix A is found to be

$$A = \frac{1}{\sqrt{2}}\begin{bmatrix}1\\1\end{bmatrix}[1 \quad 1]\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}\begin{bmatrix}-1\\1\end{bmatrix}[-1 \quad 1]\frac{1}{\sqrt{2}}$$

$$= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

and the system matrix in Eq. (10.2) is seen to be the identity matrix: Referring to Eq. (3.6) again. the system matrix for the original system may be written

$$\tilde{N}KN = \sum_{i=1}^{4} (\tilde{N}_iK_iN_i) + \tilde{N}_5K_5N_5$$

$$= (\tilde{N}KN)_{\text{torn}} + \tilde{N}_5K_5N_5 \qquad (10.3)$$

where

$$N_5 = [-[1, 0] \quad [1, 0]] \qquad \text{and} \qquad K_5 = 1.$$

Equation (IO. I) can now be used to find $(\tilde{N}KN)^{-1}$ by identifying

$$B = (\tilde{N}KN)_{\text{torn}} = I$$
$$S = -1 \qquad (10.4)$$
$$V = U = \tilde{N}_5$$

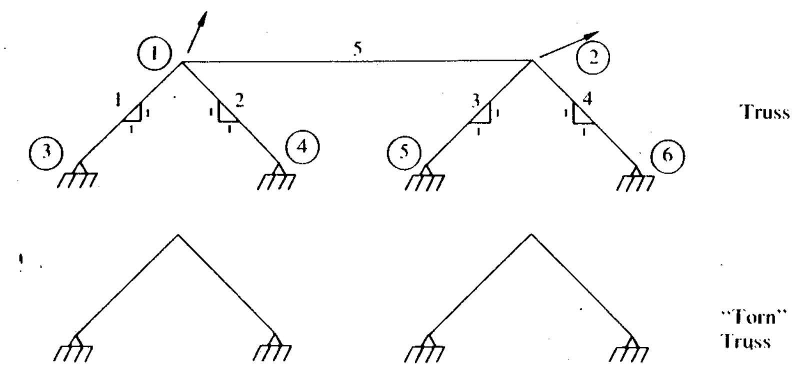

Fig. 10.1.

after which it follows from Eq. (IO. I) that

$$(\tilde{N}KN)^{-1} = I + I\tilde{N}_5\left(-1 - [-1 \quad 0 \quad 1 \quad 0]\begin{bmatrix}-1\\0\\1\\0\end{bmatrix}\right)^{-1}N_5I$$

$$= \begin{bmatrix} \frac{2}{3} & 0 & \frac{1}{3} & 0 \\ 0 & 1 & 0 & 0 \\ \frac{1}{3} & 0 & \frac{2}{3} & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Another application of Eq. (IO. I) called "doubling" is shown schematically in Fig. 10.2. The idea here is to tear a structure into a number of independent identical pieces and interconnect them using a number of steps. each of which doubles the size of the torn structure. until the original structure is achieved.

Anticipating Appendix A.5 a little. it may be noted that one of the shortcomings of tearing is that the inverse system matrix, or at least a large part of it. is constructed while Appendix A.5 shows that this is generally inefficient. It may also be noted that efficient application of tearing requires structures which are highly "symmetric." a situation which is more common in networks than structures.

10.3 K-PARTITIONING

Given'a system (structure) in the form

$$A\delta = P. \qquad (10.5)$$

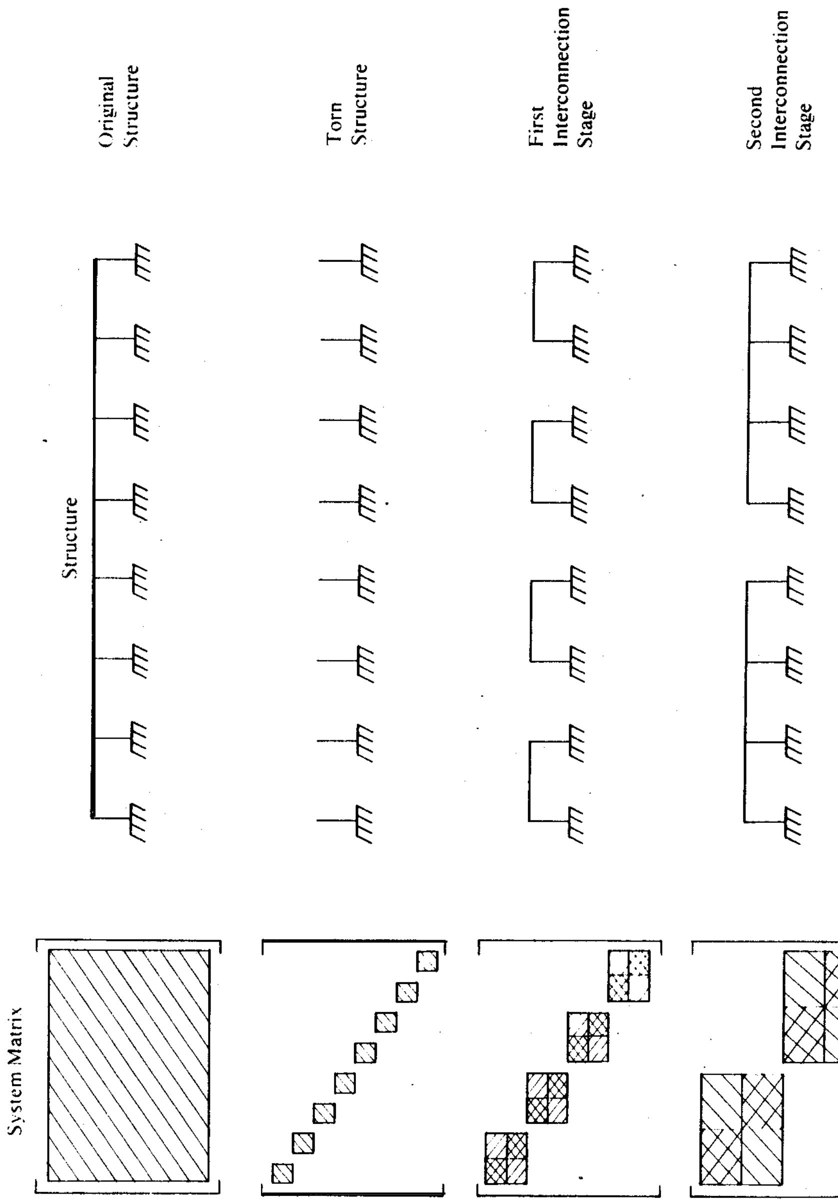

Fig. 10.2. Doubling.

rather than solving it as

$$\delta = A^{-1}P. \tag{10.6}$$

the system can be partitioned into

$$\begin{bmatrix} A_1 & A_2 \\ A_3 & A_4 \end{bmatrix}\begin{bmatrix} \delta_1 \\ \delta_2 \end{bmatrix} = \begin{bmatrix} P_1 \\ P_2 \end{bmatrix}, \tag{10.7}$$

and solved as

$$\delta_2 = (A_4 - A_3A_1^{-1}A_2)^{-1}(P_2 - A_3A_1^{-1}P_1), \tag{10.8}$$

and

$$\delta_1 = A_1^{-1}(P_1 - A_2\delta_2).$$

Obviously. this is just a matrix form of Gaussian elimination.

As with all methods for solution. the question of generality or how to proceed in the case of an arbitrary system must be considered. For example, into how many pieces should the system matrix be partitioned'? In what order should the elimination be performed'?. . . Some of these problems are dealt with in Appendix AS. some remain open questions.

10.4 METHOD OF SUBSTRUCTURES

It is sometimes convenient to subdivide a structure into its component parts for analysis. While this is not always easy to motivate from the point of view of computational effort, for practical reasons it may be desirable to do so in order to. e.g.. allow design groups to proceed almost independently on their respective parts. In this section it will be shown that the mechanics of using substructures is simply the mechanics of K-partitioning or matrix Gaussian elimination.

Figure 10.3 shows schematically an aircraft which has been subdivided

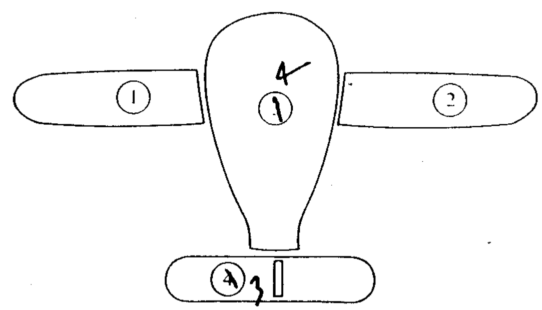

Fig. 10.3. Method of substructures