EIGHT

The Mesh Method

8.1 INTRODUCTION

The force method. an alternative form of the mesh method to which this chapter is devoted. was until recently the most popular method of structural analysis. While the decline of the force method can only be attributed to programming difficulties. it retains certain important advantages which on occasion overshadow these difficulties. The principal advantage of the mesh method is its ability to tolerate rigid elements which cause numerical difficulties (ill-conditioning) in the node method; this situation can be compared with the ease by which hinges (see Chapter 7) which cause problems in the mesh method. can be included in the node method.

The mesh method for *networks* (*see* Chapter 1) is purely topological and quite elegant. It is based upon the idea that rather than writing equilibrium equations in terms of node quantities, it is possible to combine solutions (mesh flows). each of which satisfies equilibrium. to satisfy the mesh law.

The situation for structures is basically the same as that of networks but much more complicated. In structures it is not sufficient to know connectivity, geometry must be considered and it is here that the difficulties lie. The magnitude of these difficulties becomes apparent when the generalization of the branch-mesh matrix is generated in this chapter. While N, the generalization of the branch-node matrix has occasionally become complicated. it has nevertheless always been possible to give a rather simple description of it. On the other hand. the generalization of the branch-mesh matrix which is used in Program P.5 is more complicated by an order of magnitude and is prescribed in a constructive manner.

In ‘the next section a rather formal derivation of the mesh method is given in terms of the node method. There, ideas of the degree of statical indeterminacy and compatibility are introduced in the by now familiar notation of the node method. This section is followed by an attempt to discuss simply the physics of the mesh method. Finally, a general statement of the mesh method is presented and followed by exercises.

8.2 THE MESH METHOD DERIVED’FROM THE NODE METHOD

In this section the equations of the node method are rewritten to obtain a formal statement of the mesh method. Recalling the equations. of the node method,

$$\tilde{N}F = P \qquad F = KA \qquad A = N\delta,$$

for a stable structure they can always be partitioned into

$$[\tilde{N}_T \tilde{N}_L]\begin{bmatrix} F_T \\ F_L \end{bmatrix} = P,$$

$$\begin{bmatrix} F_T \\ F_L \end{bmatrix} = \begin{bmatrix} K_T & 0 \\ 0 & K_L \end{bmatrix}\begin{bmatrix} \Delta_T \\ \Delta_L \end{bmatrix} \tag{8.1}$$

$$\begin{bmatrix} \Delta_T \\ \Delta_L \end{bmatrix} = \begin{bmatrix} N_T \\ N_L \end{bmatrix}\delta$$

after perhaps an interchange of rows, so that the matrix N_T is square and non-singular (i.e. N_T^{-1} exists). The elements of F_L constitute what is usually referred to as a set of redundants. In fact the number of independent elements in F_L is called the *degree of statical indeterminacy*. This is made more clear by writing the equilibrium equations as

$$F_T = \tilde{N}_T^{-1}(P - \tilde{N}_L F_L) \tag{8.2}$$

from which it follows that equilibrium is satisfied for arbitrary F_L provided that F_T is computed using Eq. (8.2). From the last of Eqs. (8.1) it follows that

$$\delta = N_T^{-1}\Delta_T \qquad \text{and} \qquad A,. = N_L\delta = N_L N_T^{-1}\Delta_T. \tag{8.3}$$

The last of Eq. (8.3) relates the member displacements A,, associated with redundant bars to the member displacements Δ_T of the other bars. This *compatibility equation* simply states that once Δ_T has been specified, Δ_L is also determined if the bars are to fit together.

The compatibility equation is the basis for the mesh method which follows directly once the member displacements are written in terms of

the matrix F_L, and the joint load matrix P. This is achieved simply by writing

$$A,. = N_L N_T^{-1}\Delta_T \;\rightarrow\; K_L^{-1}F_L = N_L N_T^{-1}K_T^{-1}\tilde{N}_T^{-1}(P - \tilde{N}_L F_L)$$

(using **Eq.** (8.1))

or

$$(K_L^{-1} + N_L N_T^{-1}K_T^{-1}\tilde{N}_T^{-1}\tilde{N}_L)\,F_L = N_L N_T^{-1}K_T^{-1}\tilde{N}_T^{-1}P. \tag{8.4}$$

Equation (8.4) is the equation of the mesh method; however, the form indicated is not at all convenient for computation since it requires the inversion of N_T. In a later section a more convenient form is developed and the remainder of this section is devoted to a modified form of Eq. (8.4) motivated by the topological network problem. After the topological branch-mesh matrix, let

$$C = \begin{bmatrix} C_T \\ C_L \end{bmatrix} = \begin{bmatrix} -\tilde{N}_T^{-1} & \tilde{N}_L \\ & I \end{bmatrix}, \tag{8.5}$$

in terms of which the compatibility equation, Eq. (8.3), becomes

$$\tilde{C}\Delta = 0. \tag{8.6}$$

Following networks a little further, F can be decomposed into

$$F = \mathscr{F} + f, \tag{8.7}$$

where $\mathscr{F}$ is assumed to be known and to have the property

$$\tilde{N}\mathscr{F} = P, \tag{8.8}$$

so that

$$\tilde{N}f = 0 \tag{8.9}$$

and the problem (in terms off) has been reduced to one for which there are no applied node forces. The decomposition, Eq. (8.7), is not unique and can be done in many ways, one of which is to let

$$\mathscr{F} = \Theta P, \qquad \text{where} \qquad \Theta = \begin{bmatrix} \tilde{N}_T^{-1} \\ 0 \end{bmatrix}, \tag{8.10}$$

from which it follows that

$$f = CF,,. \tag{8.1 I}$$

In terms of these variables it follows that

$$f = CF_L = F - \mathscr{F} = F - \Theta P \tag{8.12}$$

or

$$F = CF,, + OP. \tag{8.13}$$

Formally the equations of the mesh method are then

$$\tilde{C}\Delta = 0 \qquad \text{(mesh law)}$$
$$A = K\text{-‘}F \qquad \text{(Hooke’s law)} \qquad (8.14)$$
$$F = CF_L + \Theta P \qquad \text{(branch force-mesh force relationship)}$$

from which it follows **that**

$$\tilde{C}\Delta = 0 \;\rightarrow\; \tilde{C}K^{-1}F = 0 \;\rightarrow\; \tilde{C}K^{-1}(CF_L + \Theta P) = 0, \qquad (8.15)$$

or

$$F_L = -(\tilde{C}K^{-1}C)^{-1}\tilde{C}K^{-1}\Theta P, \qquad (8.16)$$

which is identical to Eq. (8.49.

8.3 THE MESH LAW

In Section 8.2 the mesh method has been derived directly from the node method, It is however most common to postulate a “mesh law” for a system from which the mesh method can be obtained directly. That is done in this. chapter for skeletal structures which have well behaved primitive flexibilities (i.e. $|K_i^{-1}| \neq 0$).

The mesh law in this case might be stated, “The relative displacement of two adjacent points as computed by moving around any closed loop must be zero.” The idea here is that given member displacements, it is possible to “fix” a reference joint and then go from joint to joint through the structure computing displacements. In particular, the computed displacement of the reference point as determined by moving around any path must be zero. For stable structures the number of independent conditions which can be obtained in this fashion equals the degree of statical indeterminacy of the structure.

Figure 8.1 shows a simple structure to which this process is applied. It is convenient to use Eq. (5.2) in the form

$$\Delta_i = \eta_i^{+}\delta_A + \eta_i^{-}\delta_C. \qquad (8.17)$$

Starting at joint 1,

I. Let $\delta_1 = 0$
2. Compute $\delta_2 = (\eta_1^{+})^{-1}\Delta_1$
3. Compute $\delta_3 = (\eta_2^{+})^{-1}[A, - \eta_2^{-}\delta_2]$
$= (\eta_2^{+})^{-1}[\Delta_2 - \eta_2^{-}(\eta_1^{+})^{-1}\Delta_1]$
4. Compute $\delta_1 = (\eta_3^{-})^{-1}[\Delta_3 - \eta_3^{+}\delta_3]$
$= (\eta_3^{-})^{-1}[\Delta_3 - \eta_3^{+}(\eta_2^{+})^{-1}\{\Delta_2 - \eta_2^{-}(\eta_1^{+})^{-1}\Delta_1\}]$
$= 0$

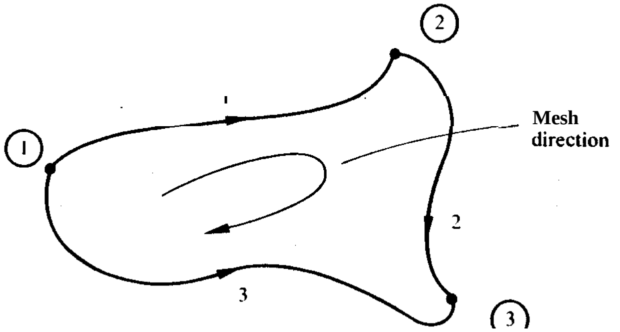

Fig. 8.1. A single mesh frame.

Collecting terms gives the compatibility equation

$$(\eta_3^{-})^{-1}\eta_3^{+}(\eta_2^{+})^{-1}\eta_2^{-}(\eta_1^{+})^{-1}\Delta_1 - (\eta_3^{-})^{-1}\eta_3^{+}(\eta_2^{+})^{-1}\Delta_2 + (\eta_3^{-})^{-1}\Delta_3 = 0. \qquad (8.18)$$

An equation like this can be written for any closed loop (mesh) within a structure. It is of some interest to compare this equation to Eq. (1. I 1) for networks.

Meshes also play an important role with regard to force systems. In particular, it is possible to construct force systems which satisfy equilibrium for an unloaded structure ($P = 0$) by selecting arbitrarily the force in one member of a mesh and then proceeding from member to member around the mesh computing member forces. It is convenient to use Fig. 8.1 again to demonstrate this procedure. But first write Eq. (5. I) in the form.

$$\tilde{R}_i f_i^{+} = \tilde{R}_i\tilde{N}_i^{+}F_i = \tilde{\eta}_i^{+}F_i \quad \text{and} \quad \tilde{R}_i f_i^{-} = \tilde{R}_i\tilde{N}_i^{-}F_i = \tilde{\eta}_i^{-}F_i. \qquad (8.19)$$

Starting with member 1,

I. Let F_1 be arbitrary
2. Compute $\tilde{R}_1 f_1^{+} = \tilde{\eta}_1^{+}F_1$
3. But $\tilde{R}_1 f_1^{+} = -\tilde{R}_2 f_2^{-}$ (equilibrium)
4. Therefore $F_2 = (\tilde{\eta}_2^{-})^{-1}(\tilde{R}_2 f_2^{-}) = -(\tilde{\eta}_2^{-})^{-1}\tilde{\eta}_1^{+}F_1$
5. Compute $\tilde{R}_2 f_2^{+} = \tilde{\eta}_2^{+}F_2$
6. But $-\tilde{R}_2 f_2^{+} = \tilde{R}_3 f_3^{+}$ (equilibrium)
7. Therefore $F_3 = -(\tilde{\eta}_3^{+})^{-1}(\tilde{R}_3 f_3^{+}) = (\tilde{\eta}_3^{+})^{-1}\tilde{\eta}_2^{+}(\tilde{\eta}_2^{-})^{-1}\tilde{\eta}_1^{+}F_1$

Note that if Eq. (8.18), the compatibility equation, is cleared so that the coefficient of A, is I, the coefficients above which relate F_1 to the member forces become identical with the transpose of the coefficients which occur in the compatibility equation.

It should be clear now that a set of member forces which satisfy equilibrium with respect to the loaded structure can be found by first reducing the structure to a tree and then moving inward from the tree tips computing member forces as has been done in the preceding paragraph but using the appropriate nonhomogeneous equilibrium equations.

8.4 AN ALTERNATIVE FORMULATION

There is one aspect of the formulation of Section 8.2 which is not completely general. It is the fact that the lower part of the matrix C is taken to be the identity matrix and it is associated with the fact that it is possible to identify each element of F_L with a branch force. (In graph theory this is equivalent to identifying meshes through tree links.) In this section a formulation is presented which does not include such a restriction-on C and which is practically motivated by numerical considerations. Briefly, the number of non-zero terms in the system matrix of the mesh method can be reduced by using the more general form of C; this more general form' is used in Program P.5.

In the alternative formulation Eq. (8.13) is replaced by node

$$F = \mathscr{C}F_M + \bar{F}. \tag{8.20}$$

$\bar{F}$ Here F is any set of forces which satisfy equilibrium,

$$\tilde{N}\bar{F} = P, \tag{8.21}$$

and $\mathscr{C}F_M$ represent independent force systems which satisfy equilibrium with regard to the unloaded structure. It is assumed that the number of elements in the "mesh force matrix" F_M equals the degree of statical indeterminacy of the structure. Defining $\mathscr{C}$ through the use of Eq. (8.20) rather than using Eq. (8.13) is equivalent to allowing any mesh description for a graph rather than being restricted to link-tree description of meshes. In view of equilibrium. it follows that

$$\tilde{N}(\mathscr{C}F_M) = 0 \quad \rightarrow \quad \tilde{N}\mathscr{C} = 0, \tag{8.22}$$

and the mesh method follows

$$\tilde{\mathscr{C}}\Delta = 0 \quad \rightarrow \quad \tilde{\mathscr{C}}K^{-1}F = 0 \quad \rightarrow \quad \tilde{\mathscr{C}}K^{-1}(\mathscr{C}F_M + \bar{F}) = 0$$

or

$$\tilde{\mathscr{C}}K^{-1}\mathscr{C}F_M = -\tilde{\mathscr{C}}K^{-1}\bar{F} \tag{8.23}$$

as before.

In conclusion it is perhaps worthwhile to repeat an earlier remark. It is only possible to relate the matrices F_L and F_M to topological quantities (meshes) for structures without hinges (in general releases). For trusses and other structures which contain releases, the mesh method must be approached by examining the rank of the matrix N as was done in Section 8.2.

8.5 EXERCISES

1. Modify the Program P.5 to generate its own mesh descriptions rather than reading them in as input.
2. Modify Program P.5 to include the effects of variable moment of inertia, member load, and temperature.