3. Indicate the modifications required in the formulation when an end of member force is used as the member force (e.g. when $F_i = f_i^+$).
4. Modify computer Program P.3 to perform plastic analysis.

Answer: In this. exercise, the simplest plastic analysis problem is considered in which:

1. Loads are only allowed to act at nodes.
2. All supports are fixed.
3. The reduction of the moment capacity of a beam due to the presence of axial load is neglected.
4. The ‘structure is subjected to proportional loading. i.e. $P = \lambda P_0$, where P is the usual joint load matrix; P_0 is a matrix which represents the fixed ratios of the loads, and $\lambda > 0$ is a scalar.
5. The members are uniform between nodes.

For this problem ***(see** P. Hodge, **Plastic Analysis of Structures.*** McGraw-Hill. 1959). one possible formulation is through ***linear programming:***

Maximize A

Subject to:

$$\tilde{N}F = \lambda P_0 \quad \text{(equilibrium)}$$
$$|m_i^+| \leqslant \mu_i \quad \text{(the bending moment}$$
$$|m_i^-| \leqslant \mu_i \quad \text{diagram must be safe)}$$

where

μ_i — capacity of member i

In this formulation the collapse load is the largest for which it is possible to satisfy equilibrium and not exceed the capacity of any member.

The details of the solution of this problem are contained in computer Program P.8. This exercise indicates that once the capability of writing the equilibrium equations for an arbitrary structure is achieved. also achieved are concomitant capabilities which go beyond simple elastic analysis.

FIVE

The Node Method for Space Frames

5.1 THE SPACE FRAME

The space frame constitutes the final step of increasing complexity in class of structures made in this book. It introduces no new concepts over the plane frame. only complications. For that reason this chapter might well be omitted from an elementary undergraduate course.

‘A space frame is defined here to be a skeletal structure constructed by joining with rigid connections elements which are arbitrary curved beams. Again, the rigid connection of elements will be seen to imply no restrictions upon the generality of the formulation. and only loads which are applied to joints will be considered.

Briefly. the space frame differs from the plane frame only in dimension:

1. The displacement vector associated with each joint has three rather than two components.
2. The rotation vector associated with each joint has three components rather than one.
3. There are. therefore. six equilibrium equations associated with each joint rather than three.
4. The member displacement matrix has six rather than three components.

Note finally that corresponding to each of the above statements concerned with displacements is a statement concerned with forces and that these force and displacement comments have rather obvious ramifications with regard to the matrices N and ***K.***

Since no new concepts are developed. this chapter moves along rather quickly and is not easily taken without the background developed in the preceding chapters.

5.2 A FORMAL DESCRIPTION OF THE SPACE FRAME

Figure 5.1 shows a typical frame joint associated with which are an applied force vector $\mathbf{P}_i$, an applied moment vector $\mathbf{m}_i$, a displacement vector $\boldsymbol{\delta}_i$, and a rotation vector $\boldsymbol{\theta}_i$. These are again described for the entire structure by the matrices

$$P = \begin{bmatrix} P_1 \\ P_2 \\ \vdots \\ P_J \end{bmatrix} \quad \text{and} \quad \delta = \begin{bmatrix} \delta_1 \\ \delta_2 \\ \vdots \\ \delta_J \end{bmatrix},$$

but here the elements are

$$\delta_i = \begin{bmatrix} (\boldsymbol{\delta}_i)_x \\ (\boldsymbol{\delta}_i)_y \\ (\boldsymbol{\delta}_i)_z \\ (\boldsymbol{\theta}_i)_x \\ (\boldsymbol{\theta}_i)_y \\ (\boldsymbol{\theta}_i)_z \end{bmatrix} \quad \text{and} \quad P_i = \begin{bmatrix} (\mathbf{P}_i)_x \\ (\mathbf{P}_i)_y \\ (\mathbf{P}_i)_z \\ (\mathbf{m}_i)_x \\ (\mathbf{m}_i)_y \\ (\mathbf{m}_i)_z \end{bmatrix};$$

J again is the number of movable joints.

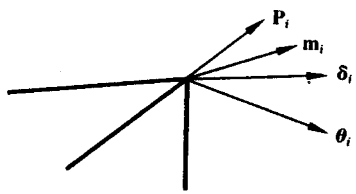

Fig. 5.1. The ith point.

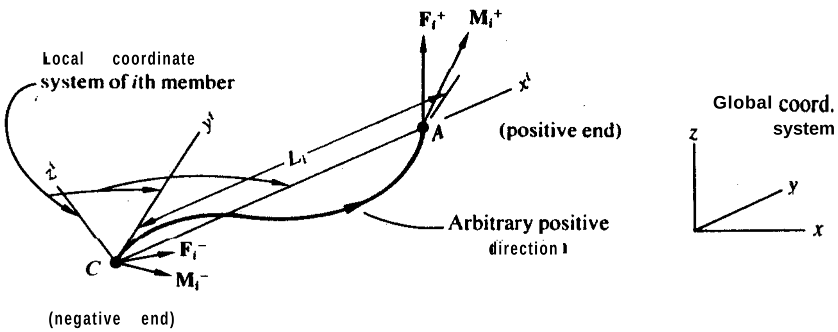

Fig. 5.2. The ith member.

The member description of a beam in space is considerably more complicated than that of a plane beam and for that reason it is shown rather schematically in Fig. 5.2. In its own local coordinate system, each member is again oriented along the x axis.

In space, a beam is acted upon by a force vector and a moment vector at each end. a total of twelve quantities. But since these twelve quantities are related by the six equilibrium equations of a rigid body in space, there are six quantities which may be specified arbitrarily. By this argument it is determined that the member force matrix will contain six components for each member in the structure. or

$$F = \begin{bmatrix} F_1 \\ F_2 \\ \vdots \\ F_B \end{bmatrix} \quad \text{and} \quad F_i = \begin{bmatrix} (\mathbf{F}_i^+)_{x^i} \\ (\mathbf{M}_i^+)_x \\ (\mathbf{M}_i^+)_y \\ (\mathbf{M}_i^+)_{z^i} \\ (\mathbf{M}_i^-)_{y^i} \\ (\mathbf{M}_i^-)_{z^i} \end{bmatrix}$$

in which the subscripts x^i, y^i, z^i indicate components in the localcoordinate system of the ith member. There is an arbitrariness in the selection of the components, of F_i here, just as there is in the case of the plane frame. (The reader is referred to the comments given in Section 4.3.) It may be noted that the components of F_i can be classified roughly as a thrust, a twisting moment. and four bending moments.

Let

$$f_i^+ = \begin{bmatrix} (\mathbf{F}_i^+)_{x^i} \\ (\mathbf{F}_i^+)_{y^i} \\ (\mathbf{F}_i^+)_{z^i} \\ (\mathbf{M}_i^+)_{x^i} \\ (\mathbf{M}_i^+)_{y^i} \\ (\mathbf{M}_i^+)_{z^i} \end{bmatrix} \quad \text{and} \quad f_i^- = \begin{bmatrix} (\mathbf{F}_i^-)_{x^i} \\ (\mathbf{F}_i^-)_{y^i} \\ (\mathbf{F}_i^-)_{z^i} \\ (\mathbf{M}_i^-)_{x^i} \\ (\mathbf{M}_i^-)_{y^i} \\ (\mathbf{M}_i^-)_{z^i} \end{bmatrix}$$

represent the end of member forces at the positive and negative ends of member i respectively. Using equilibrium they can be written in terms of the member force. F_i, as

$$f_i^+ = \tilde{N}_i^+ F_i \quad \text{and} \quad f_i^- = \tilde{N}_i^- F_i \tag{5.1}$$

for which

$$N_i^+ = \begin{bmatrix} 1 & & & & & \\ & & & 1 & & \\ & & L_i^{-1} & & 1 & \\ & -L_i^{-1} & & & & 1 \\ & & L_i^{-1} & & & \\ & -L_i^{-1} & & & & \end{bmatrix}$$

and

$$N_i^- = \begin{bmatrix} -1 & & & & & \\ & & & -1 & & \\ & & -L_i^{-1} & & & \\ & L_i^{-1} & & & & \\ & & -L_i^{-1} & & 1 & \\ & L_i^{-1} & & & & 1 \end{bmatrix}.$$

The end of member forces in the global coordinate system are now obtained by multiplying f_i^+ and f_i^- on the left by the transpose of the rotation matrix

$$R_i = \begin{bmatrix} A_i & 0 \\ 0 & A_i \end{bmatrix},$$

in which A_i is the rotation matrix (*see* Appendix 1) which transforms a vector in the global coordinate system into a vector in the local coordinate system of the ith member; e.g.

$$(F_i^+)_{x^i} = A_{11}(F_i^+)_x + A_{12}(F_i^+)_y + A_{13}(F_i^+)_z.$$

Using the equilibrium equations. $NF = P$. again to define N. it follows that N is a $B \times J$ (row $\times$ column) matrix whose elements are

$$N_{ij} = \begin{cases} N_i^+ R_i & \text{if } j \text{ is the positive end of member } i \\ N_i^- R_i & \text{if } j \text{ is the negative end of member } i \\ 0 & \text{otherwise} \end{cases}$$

Since $\Delta = N\delta$. the member displacement matrix Δ is now defined and it only remains to interpret its components. From direct calculation it follows that .

$$\Delta_i = \begin{bmatrix} (\delta_A)_{x^i} - (\delta_c)_{x^i} \\ (\theta_A)_{x^i} - (\theta_c)_{x^i} \\ (\theta_A)_{y^i} + [(\delta_A)_{z^i} - (\delta_c)_{z^i}]L_i^{-1} \\ (\theta_A)_{z^i} - [(\delta_A)_{y^i} - (\delta_c)_{y^i}]L_i^{-1} \\ (\theta_c)_{y^i} + [(\delta_A)_{z^i} - (\delta_c)_{z^i}]L_i^{-1} \\ (\theta_c)_{z^i} - [(\delta_A)_{y^i} - (\delta_c)_{y^i}]L_i^{-1} \end{bmatrix}$$

where

$$\Delta = \begin{bmatrix} \Delta_1 \\ \Delta_2 \\ \vdots \\ \Delta_B \end{bmatrix}$$

or simply

$$\Delta_i = N_i^+ R_i \delta_A + N_i^- R_i \delta_c \qquad (5.2)$$

Finally. while the stiffness matrix K for arbitrary curved members is discussed in a later chapter. for uniform straight members it is simply

$$K = \begin{bmatrix} K_1 & & & \bigcirc \\ & K_2 & & \\ & & \ddots & \\ \bigcirc & & & K_B \end{bmatrix}$$

in which

$$K_i = \begin{bmatrix} \frac{A_iE}{L_i} & & & & & \\ & \frac{J_iG}{L_i} & & & & \\ & & \frac{4EI_i^Y}{L_i} & & \frac{2EI_i^Y}{L_i} & \\ & & & \frac{4EI_i^Z}{L_i} & & \frac{2EI_i^Z}{L_i} \\ & & \frac{2EI_i^Y}{L_i} & & \frac{4EI_i^Y}{L_i} & \\ & & & \frac{2EI_i^Z}{L_i} & & \frac{4EI_i^Z}{L_i} \end{bmatrix}$$

and

- L_i – length of ith member (projection on the x axis)
- A_i – area of ith member
- J_i – torsional stiffness of the ith member
- I_i^Y – bending stiffness about y axis of the ith member
- I_i^Z – bending stiffness about z axis of the ith member
- E – Young's modulus
- G – modulus of rigidity

5.3 A DECOMPOSITION

The system matrix may be decomposed into the sum of the contributions of the individual members precisely as was done in Section 4.4 for plane frames.

A computer program illustrating the remarks of this chapter is included at the end of the book.