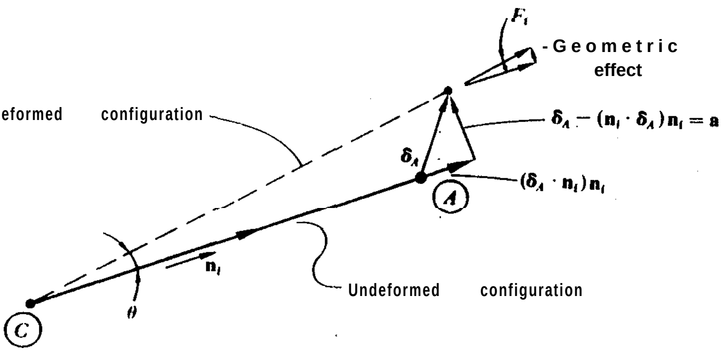

Fig. 2.8

(*see* Fig. 2.8) of small θ. In vector form. at end A this force is

$$F_i \frac{\mathbf{a}}{|\mathbf{a}|}\frac{|\mathbf{a}|}{L_i} = \frac{F_i}{L_i}[\boldsymbol{\delta}_A - (\mathbf{n}_i \cdot \boldsymbol{\delta}_A)\mathbf{n}_i]$$

it only remains to introduce this term in an appropriate manner into Programs P. I and P.2.

Probs. 1) Offset Member Stiff

2) Stiff Straight with Variable

A

THREE

A General Statement of the Node Method

3.1 INTRODUCTION

In this chapter a rather general discussion of the node method for structures is presented. The intention is to provide a formalism under which any class of structures can be considered. The reader can judge for himself the extent to which this has been accomplished by observing the wide class of problems now treated using finite elements methods (see Azar or Zienkiewicz) and the high compatibility of these methods with the formalism presented here.

With the material of this chapter, the attempt is also to unify much of this text. While Chapters 2, 4, and 5 are in fact special applications of the material presented in this chapter, Several of the developments in the other chapters depend heavily upon a clear understanding of the node method and therefore also motivate this chapter.

Before beginning the formal statement of the node method, it is perhaps in order to comment on the fact that such a large fraction of this book is devoted to it. In spite of the fact that experience indicates that the classical methods are easier to execute when working without a computer. some very large proportion of the computer programs in use today are based on the node method. The justification for this sacrifice in computational efficiency is simply that the node method is very easy to program. It is one of the delightful aspects of the state-of-the-art that it is now possible to achieve general computer programs with modest programming effort.

3.2 THE NODE METHOD

It is assumed that the structure to be analyzed has a well defined set of nodes and branches (or elements) and that two matrices are associated with the nodes.

P -joint load matrix,
δ -joint displacement matrix.

Associated with the branches (members) of the structure are

F – member force matrix,
A – member displacement matrix,
K – primitive stiffness matrix.

In terms of these quantities the equations of the node method are simply

$$\begin{aligned} \tilde{N}F &= P && \text{(joint equilibrium)} \\ F &= KA && \text{(Hooke's law)} \qquad (3.1) \\ A &= N\delta && \text{(member displacement-joint displacement equation)} \end{aligned}$$

from which it follows that

$$\tilde{N}F = P \rightarrow \tilde{N}K\Delta = P \rightarrow \tilde{N}KN\delta = P$$

or

$$\delta = (\tilde{N}KN)^{-1}P \qquad (3.2)$$

The analysis problem consists of finding, F, A, and δ given N, K, and P. Eq. (3.2) indicates that in the node method, δ is computed first from which A and F may be computed using the second and third parts of Eqs. (3.1). It should be noted that Eqs. (3.1 and 3.2) are the most simple statement of the node method; such effects as temperature, lack of fit, support settlement. etc.. will be added in a later chapter.

Nothing in this formulation implies that the structure under discussion is composed of members with two ends. Quite to the contrary, this formulation is valid beyond its use in this text and can be applied to shell and solid finite elements.

Finally. some remarks are in order concerning the “incidence matrix*‘, N. While Eqs. (3. I) simply describe a linear system. the description is peculiar in that it uses the matrix N twice. This multiple use of N is not necessary and is purely a matter of convenience which can be motivated most easily in terms of energy.

Let the work done by the external loads and the internal strain energy be defined as

$$W = \tfrac{1}{2}\tilde{P}\delta \qquad \text{and} \qquad II = \tfrac{1}{2}\tilde{F}\Delta, \qquad (3.3)$$

respectively. If the node method is to describe a “conservative system” then

$$W = E \qquad \text{or} \qquad \tilde{P}\delta = \tilde{F}\Delta. \qquad (3.4)$$

The following theorem can now be proved.

THEOREM: *If $W = E$ and $\tilde{N}F = P$ are to hold for arbitrary F, it follows that* A = $N\delta$.

Proof: Since $\tilde{N}F = P$, Eq. (3.4) can be written as

$$\tilde{F}N\delta = \tilde{F}\Delta. \qquad (3.5)$$

If Eq. (3.5) is to be valid for an arbitrary F, then A = $N\delta$. If it were not, there would exist a component of A, say A, for which

$$(N\delta)_I \neq A_I,$$

which would yield a contradiction to Eq. (3.5) by selecting the elements of F to be zero except $F_I = I$.

If some form other than Eq. (3.1) is used for the node method, it is simply necessary to use some form other than Eq. (3.3) for the energy.

3.3 THE DECOMPOSITION

With numerical computation in mind, it is worthwhile to pursue the details of the system matrix, $\tilde{N}KN$, a little further. The most striking feature of the matrices N and K is that they are sparse; this sparseness discourages their direct formation in the computer. In this section a scheme is discussed which allows the formation of $\tilde{N}KN$ from the elements of N and K without explicitly constructing the matrices themselves. It is here that use is made of the fact that each member has two ends.

The first step in the decomposition is to partition the matrix N into rows, each of which corresponds to a member. (In general these rows themselves are partitioned matrices and not single rows of elements as they were in the case of the truss.) This partitioning of N implies a partitioning of K. Since the elements (branches) of the structure are independent, the matrix K is “partitioned diagonal”. N and K therefore have the form

$$N = \begin{bmatrix} N_1 \\ N_2 \\ \vdots \\ N_B \end{bmatrix}, \qquad K = \begin{bmatrix} K_1 & & & \bigcirc \\ & K_2 & & \\ & & \ddots & \\ \bigcirc & & & K_B \end{bmatrix},$$

in which B is the number of branches (elements) in the structure. Since K is diagonal, the *system matrix* $\tilde{N}KN$ becomes a sum,

$$\tilde{N}KN = \sum_{i=1}^{b} \tilde{N}_i K_i N_i,$$

in which the term $\tilde{N}_i K_i N_i$ is the contribution of member i to the system matrix. Looking a little more closely at the structure of N and using the fact that each branch has two ends. for a member neither end of which is a support N_i has the form

$$\begin{array}{cc} & \text{col. } A \quad \text{col. } B \\ N_i = & [\cdots \eta_i^+ \cdots \eta_i^- \cdots] \end{array}$$

and

$$\tilde{N}_i K_i N_i = \begin{array}{c} \\ \\ \end{array}\left[\begin{array}{ccccc} & \text{col. } A & & \text{col. } B & \\ & | & & | & \\ \text{---} & \tilde{\eta}_i^+ K_i \eta_i^+ & \text{---} & \tilde{\eta}_i^+ K_i \eta_i^- & \text{---} \\ & | & & | & \\ \text{---} & \tilde{\eta}_i^- K_i \eta_i^+ & \text{---} & \tilde{\eta}_i^- K_i \eta_i^- & \text{---} \\ & | & & | & \end{array}\right]\begin{array}{l} \\ \\ \text{row } A \\ \\ \text{row } B \\ \\ \end{array}$$

in which A and B are the positive and negative ends of member i respectively.

It follows that. in general. each member contributes four terms to the system matrix. These terms can be placed directly into the system matrix and it is not necessary to construct N and K explicitly. The computer programs at the end of this text illustrate this procedure.

3.4 EXERCISES

1. Show how the decomposition of Section 3.3 generalizes for members with three and four nodes (triangular and quadrilateral finite elements).
2. Show how the formulation of the node method changes when the order of the components of either, but not both, of the node force matrix or the node displacement matrix is changed.

Answer: Since the truss is the only structure already discussed in detail in this text. it will be used to discuss this exercise. A more meaningful discussion would involve the selection of. e.g., member displacements for the frame.

If. for the truss. the joint displacement matrix δ were defined as it is in the preceding chapter but the joint force matrix were defined to have other components. say

$$\delta_i = \begin{bmatrix} (\boldsymbol{\delta}_i)_x \\ (\boldsymbol{\delta}_i)_y \\ (\boldsymbol{\delta}_i)_z \end{bmatrix} \quad \text{and} \quad P_i = \begin{bmatrix} (\mathbf{P}_i)_z \\ (\mathbf{P}_i)_y \\ (\mathbf{P}_i)_x \end{bmatrix},$$

it would be necessary to use a permutation matrix to maintain the physical interpretation of work and energy. This might be done using the matrix A,

$$A = \begin{bmatrix} A_1 & & & \\ & A_2 & & \bigcirc \\ & & & \\ & \bigcirc & \cdot & \\ & & & A_J \end{bmatrix},$$

in which

$$A_i = \begin{bmatrix} + & + & 1 \\ + & 1 & + \\ 1 & + & + \end{bmatrix} \quad \text{(foranyi)}$$

has been selected so that

$$\mathbf{P}_i \cdot \boldsymbol{\delta}_i = (\mathbf{P}_i)_x(\boldsymbol{\delta}_i)_x + (\mathbf{P}_i)_y(\boldsymbol{\delta}_i)_y + (\mathbf{P}_i)_z(\boldsymbol{\delta}_i)_z = \tilde{\delta}_i A_i P_i,$$

or

$$W = \tfrac{1}{2}\tilde{P}\tilde{A}\delta.$$

The node method then becomes

$$\begin{aligned} \tilde{N}F &= AP && \text{(equilibrium)} \\ F &= KA && \text{(Hooke's law)} \\ \Delta &= N\delta && \text{(member displacement-joint displacement)} \end{aligned}$$

But since $A^{-1} = \tilde{A}$. the equilibrium equation can be modified so that the equations of the node method read,

$$\begin{aligned} \tilde{B}F &= P \\ F &= KA \\ A &= N\delta \end{aligned}$$

in which

$$B = NA,$$

a formulation in which repeated use is not made of the matrix N.